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Editorial
May 27, 2013

The new double issue of Mathematics Teaching-Research Journal on line comes out at the critical time for mathematics education marked by the increased tensions accompanying introduction of the CC curriculum, its assessment and the use of assessment for the improvement of mathematics learning in the classroom. Two sources of the increase of the tension are:
doubts about the capability of the technological environment to support the assessment process (news: in Indiana, Kentucky, Minnesota, and Oklahoma [which] were linked to the states’ assessment providers: CTB/McGraw-Hill, in Indiana and Oklahoma; ACT Inc., in Kentucky; and the American Institutes for Research, in Minnesota; Education Week, Friday, May 7, 2013, Volume 32, Issue 30) and


The inadequacy of the computer based technology for the assessment has been strongly confirmed by the difficulties Hostos CC and all other community colleges of CUNY experience with their new CUNY Algebra exit exam – the acquired system Maple TA doesn’t allow for the item-by-item analysis of every classroom nor as any Community College of CUNY. It allows only the analysis on central level of the total participating cohort of 14,000+ students, eliminating thus the possibility of teaching and learning improvement in every classroom. Alerted by us to the problem last semester, the Central Office of Testing promised, to have it working for the Spring 2013 examination session, as of now, the system still is not capable to fulfill its function – the Central Testing Office promises it to be ready by the end of the May.

The issues with the preparation of the teachers to utilize the information for the development of the adaptive instruction are substantial; adaptive instruction has been described competently by the (Darro et al, 2011) document stating that: “For that [success] to happen, teachers are going to have to find ways to attend more closely and regularly to each of their students during instruction to determine where they are in their progress toward meeting the standards, and the kinds of problems they might be having along the way. Then teachers must use that information to decide what to do to help each student continue to progress, to provide students with feedback, and help them overcome their particular problems to get back on a path toward success. This is what is known as adaptive instruction and it is what practice must look like in a standards-based system.” (CPRE Report, 2011) Adaptive instruction mentioned here is closely related to “Formative assessment [which] involves a teacher in seeking evidence during instruction (evidence from student work, from classroom questions and dialog or one-on-one interviews, sometimes from using assessment tools designed specifically for the purpose) of whether students are understanding and progressing toward the goals of instruction, or whether they are having difficulties or falling off track in some way, and using that information to shape pedagogical responses designed to provide students with the feedback and experiences they may need to keep or get on track.” CPRE Report, 2011.

The same report continues: “Teachers must receive extensive training in mathematics education research on the mathematics concepts that they teach so that they can better understand the evidence in
student work (from OGAP-like probes or their mathematics program) and its implications for instruction. They need training and ongoing support to help capitalize on their mathematics program’s materials, or supplement them as evidence suggests and help make research based instructional decisions.”

At the same time, the educational research professions starts realizing that “it is the teacher who can affect to the greatest extent the achievement of one of the main purposes of the research enterprise, that is, the improvement of students’ learning of mathematics.” (Kieren et al, 2013)

Volume 6 N 1 & 2 of Mathematics Teaching-Research Journal on line contains several papers which take assessment of the situation seriously and propose several teaching innovations which may work in the process of improvement of learning. Our colleague from Switzerland, Robert Catanuto informs about concept map based technique of integrating student interests with the concepts of mathematics, our colleagues from New Zealand, Nugzar Nachkebia and Marina Alexander investigate the relationships between the key probability and informal statistical inference concepts and on that basis conduct the critique of informal statistical inference rules adopted in New Zealand high schools. The two co-editors of mtrj present the research into creativity for all students based on the theory of The Act of Creation of (Koestler, 1964)

These “front line” papers are supported by four papers describing the organizational background for that work. Collaborators centered around our seasoned contributor Rohitha Goonatilake from Texan A&M inform about the efforts and their results in the context of STEM developmental education and then first college course College Algebra, Evans Brian analyzes the degree of social awareness amongst the incoming teachers of mathematics while Sherese Mitchelle reveals “the behind” of the early childhood education.

The double issue is topped by the research paper of the young Chilean researcher doing her PhD work in France, Raquel Barrera who investigates the “mysterious” geometrical multiplication of numbers formulated by Descartes. The mysterious aspect of Descartes multiplication becomes more transparent when we realize that his technique is equivalent to the old Hindu and Chinese Rule of Three practiced at Bronx CC by our co-editor Vrunda Prabhu, recently passed away, in her developmental classes of arithmetic and algebra. Thus, for example the multiplication $\frac{1}{4} \times \frac{1}{8}$ can be seen arising from the double line proportion schema, so that $\frac{1}{4} : X = 1 : \frac{1}{8}$, what with the help of the law of “means and extremes” gives us the required multiplication.
On the Meanings of Multiplication for Different Sets of Numbers in Context of Geometrization: Descartes’ Multiplication, Mathematical Workspace and Semiotic mediation.

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Abstract
In this article, I briefly present an experimental lesson – from Descartes’ multiplication to the geometric meaning of the product of complex numbers – connecting multiplication and some of its geometric meanings. I will also present some examples of our experiments conducted in French high schools and some elements of the analysis process to identify the possibilities of connecting multiplication and some of its geometric meanings through the production of interactions between different registers of representation and a semiotic mediation within a Mathematical Work Space (MWS).

« For every question and every phenomenon, in order to follow the normal psychological route of scientific thought, we must go from an image to a geometric figure, and then from a geometric figure to an abstract form. » (Translation, Bachelard, 1938, p. 10.)
Introduction
The fact that the notion of multiplication is closely associated with the idea of calculation can impede students from imagining a geometrical representation of the product. In the same way, the association between real numbers and the notion of magnitude can also get in the way when representing negative numbers, as well as when giving meaning to the product of two negative numbers. Thus, the multiplication of negative integers does not allow a geometric representation unless the “quantities” are treated in terms of orientation and direction (Argand, 1806). The extension of the definition of the operations for complex numbers is linked to the representation of imaginary quantities by vectors or Wessel’s line segments (Flament, 2003). As a result, transformation is the only context in which multiplication and some of its geometric meanings can be connected. We are establishing a relationship between the different meanings of our mathematical object, with geometry as the glue holding them all together.

Geometry: an intermediary between a mathematical object and understanding
Geometric representations encourage the use of cognitive variables favoring the understanding of a mathematical object. The abstraction of arithmetic and algebraic concepts also stems from the fact that they are only represented through a symbolic diagram (Radford, 2003) where a sign “bears an arbitrary or non-motivated relationship to its signified” (Radford, 2003, p.5). So, in addition to these signs, it seems to us that it is always necessary to have an intermediary between a conception and access to its meanings, given that “there is not mathematical thinking without using semiotic representations” (Duval, 2008, p.1).

Consequently, two questions organized our research: could a geometric treatment using transformations allow for a link between the meanings of multiplication for different sets of numbers? In addition, will students be able to establish connections between multiplication and geometry? This organization helped us focus our research, first on epistemological investigation work, and second on experimental research.

In this article, we will show how our desire to study the understanding of multiplication in a geometric context led us to design experimental course material. We integrated these three elements – multiplication, its meanings and geometric transformations – in an original experimental situation, created to respond to the second question presented above. We will observe some examples of students’ work on this non-traditional material requiring changes of register of semiotic representation in a process of semiotic mediation.

Our theoretical framework: The Geometrical Work Space
After analyzing the notion of Mathematical Work Space (MWS) (Kuzniak, 2011), we determined that this theoretical approach could suitably account for the complexity and richness of students’ mathematical work. This notion assumes that a network has been created on two levels, one cognitive and the other epistemological. This network relies on a certain number of geneses, which can be semiotic, instrumental or discursive (cf. Figure
The analysis of this bilateral relation allowed us to expand our theoretical knowledge of mathematical work spaces and to determine the theory’s flexibility. As we will see, this flexibility allows us to combine several theories in order to analyze students’ chosen paths within an empirical mathematical working space.

Figure 1: A genetic approach to the Geometrical Work Space

The starting point for the *geneses* linking the two levels of the MWS is traditionally placed on the epistemological level: for example, the visualization of an abstract mathematical object in a real or material space can be produced by the manipulation of artifacts in the construction of a figure. Still, the components on the epistemological level can be set in motion by needs on the cognitive level. A construction with artifacts can respond to a need for demonstration; the construction of a figure in a paper-pencil environment can be the result of a visualization allowing certain properties to assume a new configuration, or it can assemble the elements necessary for a proof. Thus, within these processes, called *geneses*, we can see not only the existence but also the permanent interactions between different registers of semiotic representation: we can make the transition from proof to construction through a change in register of representation (cognitive entrance); a geometric configuration, a sign or *representamen* (epistemological entrance) can prompt a visualization (cognitive action) making use of the properties and axioms (epistemological action) leading to a proof. As Duval states, “the only way to have access to [mathematical objects] is using signs or semiotic representations” (Duval, 2006, p. 107).

Finally, we have the makings of a hypothesis: being conscious of the metaphorical meaning of a mathematical object could allow a point of entry starting at the cognitive level of an MWS, which at the same time would encourage manipulating the components of the epistemological level. The metaphorical characteristics of this mathematical object,
whose meaning we wish to construct using geometry, would therefore allow students to establish the transition between the cognitive level and the components of the epistemological level:

“Metaphors are not just rhetorical devices, but powerful cognitive tools that help us to build or grasp new concepts, as well as solving problems in efficient and friendly ways” (Soto-Andrade & Reyes-Santander, 2011, p. 2).

A combination of theories: Mathematical Work Space (MWS) and the role of sign-artifacts in a socially interactive space

All of the different studies dealing with semiotic notions present in the process of learning/teaching mathematics—whether they include information technology or not, whether or not they talk about registers of semiotic representation or pay special attention to the role of language and the understanding of mathematical objects—all of these positions “[revolve] around the relationship between mathematics and semiotics, concern questions of an epistemological, cognitive and sociocultural order” (Falcade, 2006, p. 3-4). However, within an MWS, the didactic question does not include explicit interactions between different individuals. Additionally, the MWS does not necessarily include other intermediaries, aside from the teacher, between the learners and the knowledge to be acquired or developed. It seems appropriate, then, to explicitly include mediator intermediaries where mathematics and semiotics can be found, where the different geneses, figural, discursive and instrumental, occur, at the point where semiotic mediation and, when possible, social mediation, can facilitate access to research and the acquisition of meaning of mathematical objects. That said, given our special interest for semiotic genesis within a mathematical work space, the limitations of the existing semiotic approach as well as the theoretical definition of artifacts (Kuzniak, 2004) led us to look for other theoretical approaches dealing with semiotic mediation and the social construction of mathematical knowledge. We concentrated on Bartolini Bussi and Mariotti’s (2008) work on Semiotic Mediation Theory, aspects of which we associated with Radford (2004) and Sfard’s (2008) reflections on the social construction of mathematical knowledge and the complexity of the process of understanding a mathematical object.
Figure 2: Diagram showing the dynamic aspect of the MWS’s components and the arrangement of the epistemological and cognitive levels, caused by the action of the sign-artifact in a context of semiotic mediation.

From a didactic point of view, Semiotic Mediation Theory includes elements such as the direct manipulation of tools, either in the form of concrete objects taken from the history of mathematics, or in the form of technological artifacts. The theory also considers the precise organization of work in the classroom, where the relationships between the individual dimension, work in pairs and the collective dimension all play a role, and where oral and written activities complement one another. Finally, the theory also considers students’ reading and interpretation of historical primary sources, aided by the teacher (Falcade, 2006).

The inclusion of historical and/or technological mediators as sign-artifacts on the one hand, and on the other hand the importance of collaborative work within the learning-teaching process, were the key elements that brought us to integrate Semiotic Mediation Theory and the Mathematical Work Space. Thus, we’ve included the MWS in a socio-constructivist learning process where the sociocultural and semiotic dimensions are included in the proximal development zone defined by Vygotsky (1934-1997).

THE EXPERIMENTAL LESSON: FROM DESCARTES’ MULTIPLICATION TO THE GEOMETRIC MEANING OF THE PRODUCT OF COMPLEX NUMBERS

The study of historical, epistemological and didactic research was a long journey which allowed us to characterize the relationship between geometry and arithmetic, as well as the importance of always having a representation of the algebraic and arithmetic objects in geometry. This is corroborated by the fact that a new geometry had to be designed after centuries of research.
A significant result of this study is an example of ancient Greece: Thales' theorem. In the words of Rudolf Bkouche, the heart of the relationship between the geometry and numbers can be found in this theorem. It allows us to state of the conditions of proportionality of segments, either through measurement or with the method of analytical geometry and coordinates. We need only recall the two key elements for each statement of Thales’ theorem: parallel lines and proportions and proportions and geometry (i.e. multiplication and geometry).

Through this study, we also confirmed that multiplication for different sets of numbers corresponds to a transformation in the plane:

- For rational numbers, multiplication is defined in enlargements and reductions (Brousseau, 1986).
- For integers, multiplication gained meaning when the idea of absolute magnitude was combined with the idea of direction (Argand, 1806).
- Complex numbers’ multiplication is related to the representation of imaginary quantities by vectors (Wessel, 1797).

Finally, through this initial study, we have established a link between the product of complex numbers, Descartes’ multiplication and Thales' theorem. This is because the starting point for Wessel’s representation of multiplication of "straight lines" finds its origin in Descartes’ multiplication.

**Figure 3: Parallel between Descartes’ multiplication and Wessel’s product of line segments**

Observing the geometric representation of Descartes’ multiplication, we can recognize an icon of Thales' theorem. Studying this configuration in depth, we can confirm that it is justified by Thales’ theorem, and also that the parallelism of lines and proportions are present even when we change the nature of the factors! We will later show how we integrated all these elements in the experimental part of this project.
AN OUTLINE OF OUR METHODOLOGY

Observing several students’ work on our non-traditional material allowed us to study their ways of solving problems in a mathematics lesson requiring changes of register of semiotic representation in a process of semiotic mediation.

Students in Terminale S (twelfth grade scientific track) were asked to solve a series of five questions suggesting a geometric approach to the multiplication of real and complex numbers (Appendix 1). The activity was introduced in four Terminale S classes by their teachers. Thirty-four groups of two to four students worked on the activity for two hours in class. This session was integrated into the usual series of lessons by the teachers, who had just begun a chapter on complex numbers. At first, the students were instructed to make a geometric construction of the product of two real numbers in the plane, as proposed by Descartes in his *Geometry* (1637). Next, the students had to find a relationship between the points given on a plane and the multiplication of complex numbers. The final question of the series’ (only answered by twenty-four of the thirty-four groups working on the activity) called on students to think back on the entire activity. It played a fundamental role in the exploratory process, and its analysis allowed a first description of the paths followed by students moving between Descartes’ multiplication and the understanding of the geometric meanings of multiplication for different sets of numbers. In order to describe the role of geometrization in students’ approaches to multiplication, we studied their way of solving geometric construction problems involving the multiplication of real and complex numbers. Gradually, we’ve begun forming a response to our research question (cf. Introducing the mathematical content of analysis: linking multiplication to geometry) “now transformed” and seeing it through the eyes of our main theoretical framework: are there interactions between the cognitive and epistemological levels of the MWS employed by students showing evidence of a geometric understanding of multiplication? Through this methodology, we determined students’ chosen paths between Descartes’ multiplication and the understanding of the geometric meanings of multiplication for different sets of numbers.

In this article, we outline four paths so as to illustrate some of the experiment’s results and the way we used our combination of theories to analyze students’ mathematical work. In this work we analyzed the use of previously studied mathematical content and the way students employ it; interactions produced between the components of the MWS employed by students; the role played, as a *sign-artifact*, by the configuration of Thales’ theorem corresponding to the geometric representation of Descartes’ multiplication; the identification the *origin of geneses in a geometrical work space*. The mediating sign-artifact is therefore an essential element of our didactic proposals. As we will see, our artifact is a sign: a mathematical sign, a geometric representation and an icon of Thales’ theorem. Recognized by students in the first question of the series as a tool to be employed in a proof, the sign must evolve throughout the collaborative lesson. The goal of its systematic use in the activities is the collective production of new signs, which correspond to new interpretations of the same artifact. We can associate this last point, especially concerning the evolution of signs and the way they influence discourse, with
what Anna Sfard calls a “visual mediation.” This is the place where “visual mediators have been defined as providers of the images with which discursants identify the object of their talk and coordinate their communication” (Sfard, 2008, p.147). We hope that our theoretical combination will allow us to study whether geometric representations, either given to or produced by the student, recognized as psychological tools or sign-artifacts, are capable of producing a precise mathematical object—in this case, multiplication.

STUDENTS’ PATHS: ANALYZING A FEW RESULTS THROUGH THE EYES OF OUR THEORETICAL COMBINATION

The groups taking part in our experiment were initially classified by hand (i.e. we studied each sequence and each response to the final question, looking for elements of an answer that either corresponded to or differed from our initial determination) according to their responses to the last question of the series, leading to a first classification with three possible types of responses: Transformation (T); Proportionality and Thales’ theorem (PTTh); complex (C), without explicit references to geometric transformations. The determination of students’ paths was thus based on an analysis of the entire process leading them to their response to the last question. However, as I already said, some groupss did not answer the final question. Here I will present the work done by two of them because of their significant responses to the other questions.

Comparison between two groups showing different conclusions: Proportionality and Thales’ theorem (PTTh) and Transformation (T).

The two groups were initially given two different classifications. We will examine the groups’ differences beyond certain similarities in their responses to the final question. Group C1-I9 (T) bases its conclusion on an immediate visual connection between multiplication and the sign rule as seen in the Cartesian plane. Then the group extends this relation to Descartes’ multiplication as well as any multiplication with any type of factor. The most striking observation we can make about their response, and which shows the close links between figural and discursive geneses, is the connection made between the sign rule and vectors’ angles. The change of frames related to the sign rule is due to our activity, because the geometric manifestation of the sign rule does not appear in the French curriculum. In order to reinforce this idea, we emphasize that the group specifically identified the nature of the angles, especially the zero angle and the flat angle which allow a connection between real and complex numbers in this geometric configuration.

Group C3-I1 (PTTh) arrives at a conclusion that takes into account the different parts of the lesson. They show the properties of multiplication of real and complex numbers, justifying them with Thales’ theorem.
They present their geometric interpretation of multiplication of real and complex numbers as a generalization of Thales’ theorem in the Cartesian plane and then in the complex plane. Can we say that the word “generalization” clearly accounts for a connection between the different aspects of the lesson? In a way, yes, because the lesson requires students to extend different sets of numbers.

For a more complete view, the figures above show the complete responses of both groups to question 4.b. It seems clear that the groups’ algebraic knowledge of complex numbers guided their geometric construction, which shows no connection with the lesson’s previous constructions. The algebraic properties of the multiplication of complex numbers are already part of students’ theoretical references and they orient the students’ construction, which is correct but isolated from the rest of the lesson. In their responses, nothing explicitly accounts for the visualization and geometric comprehension of complex numbers as a transformation in the plane. The entrance into the MWS for this question is epistemological, then, resulting in a construction with a ruler allowing the
students to visualize the placement of the product of two complex numbers in the plane. In this lesson, the two groups take different approaches to question 2b. Here, only group C1-I9 (T) has used transformations in its response to one of the activity's first questions.

Figure 6: Above, group C3-I1’s response to question 2.b. Below, response 2.b for group C1-I9

In analyzing this response, it is quite interesting that the students refer to the reduction and enlargement of triangle BCA. The relationship of proportionality implied in Thales’ Theorem and the geometric representation of Descartes’ multiplication were not approached according to the segments-factors and the segment-product representing the proportionality. The visualization of similar triangles favors the immediate use of Thales’ theorem because the students interpret this reduction-enlargement using their already available knowledge. These paths present their own specific characteristics. The “T” group mentions transformations quite early, in the second question, and their conclusion (final response) is very rich, relating the sign rule and geometry. The group using Thales produces a response describing multiplication for different sets of numbers but does not explicitly link the icon of Thales to a geometric representation of multiplication for different sets of numbers.

Synthesis of the analysis

<table>
<thead>
<tr>
<th>Variable</th>
<th>Analysis results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entry into the MWS</td>
<td>Mixed but largely epistemological.</td>
</tr>
<tr>
<td>Semiotic genesis/links between different registers of representation</td>
<td>Semiotic genesis of unknown origin, especially in the response given by C1-I9 to the second question. A semiotic genesis of cognitive origin may have occurred with the visualization of similar triangles, followed by the visualization of a transformation (reduction-enlargement) of these triangles through multiplication. A significant link between registers of representation was made during the association between the sign rule and the representation of the product of a positive and a negative number in the “affine” plane (C3-I1).</td>
</tr>
<tr>
<td>Semiotic mediation of the sign-artifact</td>
<td>The action and evolution of the sign-artifact were identified thanks to a specific explanation of the existence of a zero angle in Descartes’</td>
</tr>
</tbody>
</table>
sign-artifact product (which was necessarily transposed into the “affine” plane). A link was therefore produced between the properties of the icon of Thales’ theorem and the properties of multiplication of complex numbers.

Geometrical meaning of multiplication Hypothetically, the product was interpreted as resulting from a transformation in the plane. This could have been a possible interpretation of the sign rule in terms of angles and by generalizing the meaning of any product to the product of two complex numbers.

Comparison between two groups not having answered the last question: group A (C1-I13) and group B (C1-I14)
The responses of these two groups rarely corresponded to the answers that we have identified as the most frequent, but they coincided in the first two questions. Group C1-I14 did not answer question 3.b.

Figure 7: Response to question 3.b for C1-I13
We already know that some answers are significant and their analysis has allowed us to understand, for some paths, what happens in the answers to the last two questions. This is the case of the answers to questions 2.b and 3.

Figure 8: Left, group A (C1-I13) response to question 4.a. Right, response 4.a for group B (C1-I14)
In response to question 3, we found that group A gave an answer by referring to the position of the product depending on the nature (negative integer) of a factor. This response is significant inasmuch as it may influence students’ answers to other questions in sequence, as we’ve seen with other paths. Group B developed its response to the draft following the set of orthogonal projections and the suggested technique of two applications Thales’ theorem. However, the group did not write an answer for this
question.

From what we see in the answers to question 4.a, both groups take into account the geometric properties of the multiplication of complex numbers, which they have derived from the way the geometric data corresponds to the products represented previously, or which they might have already known before our experimental session. These theoretical considerations are important because these responses could in some way influence the theoretical discourse allowing students to answer question 4.b. Knowing these properties could easily enable students to establish a technique leading to the construction of the product of two complex numbers among the most common responses. This construction, from knowledge of the properties of complex numbers in algebraic language, could account for students' ability to represent the complex product in another register of representation. As we have already seen in the paths studied previously, the power of the theoretical discourse conditioned student responses without allowing them to act as expected: they should have considered the first part of the questionnaire, recognizing the icon of Thales' theorem as a representation of multiplication and its geometric meanings, including transformations.

Therefore, the opportunity to represent the same object in different mathematical registers of representation does not necessarily mean that students understand the meaning of this mathematical object. This is the cognitive complexity that we must take into account when we assign activities where students must implement a conversion process: we need to take into account that the mere change of register of semiotic representation does not involve the reification a mathematical object.
At this stage of the study the paths of students who did not respond to the last question, I present the surprising answers of groups A and B to question 4.b where they had to build the product of two complex numbers. In the answer given by C1-I13, measurements and calculations taking into account the unit suggest to us that the construction applies the theoretical discourse on the multiplication of modules, already studied before this experiment. However, as we have already said, this response could also be justified by Thales' theorem, which has been used throughout the sequence. Nevertheless, even with some flaws related to length measurements, students’ constructions (in terms of morphological similarity to geometric representation of Descartes’ product and the proper placement of points U, A and E) could result from similitude and reflect a geometric understanding of the meaning of the product. Students were able to go from realizing that Descartes’ product has a zero angle to understanding the complex product: the power of Thales' theorem can still prove the product by rotation (adding angles) to the lengths’ multiplication. Similarly, group C1-I14 shows in its response that it is well aware of the possible failure to consider the theoretical discourse on geometric properties of multiplication of complex numbers: their construction shows an interpretation of Descartes’ product in terms of similar triangles.

This consideration may stem from the fact that Descartes’ configuration could be interpreted as being composed of two similar triangles, which result from an enlargement.
in the first question and a reduction in the second one. In this case, the enlargement resulting from a homothety is also accompanied by a rotation. This can be identified by corresponding angles in the figure. Finally, since there is no other oral or written information that allows us to access the work done by these groups in greater detail, we are left to speculate as to students’ motivations for responding in ways such as we have described. It would have been very interesting to see the way in which these particular groups could express themselves in response to the last question.

We note finally that our results are still ambiguous and that our data obtained from students is too sparse to be interpreted as facts. In this case, we would have a tangible effect or influence caused by our sign-artifact. Nevertheless, these results are far from being insignificant. In addition, research on students’ paths based on answers or non-answers to the last question oriented our data analysis and we have highlighted differences and similarities between these paths. However, we need additional information to further develop our analysis. We return to this issue in our conclusions.

Synthesis of the analysis

<table>
<thead>
<tr>
<th>Variable</th>
<th>Analysis results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entry into the MWS</td>
<td>Mixed. Cognitive entries are hypothetical but generally possible.</td>
</tr>
<tr>
<td>Semiotic genesis/links between different registers of representation</td>
<td>A semiotic genesis could allow students to go from the algebraic expression of complex multiplication to the construction of the product; we would say that this is an epistemological genesis. Nevertheless, taking into account the constructions made in answers to question 4.b, this genesis may have its origin on the cognitive level. The properties of the product of complex numbers were interpreted in terms of transformations. This leads the group to implement an instrumental genesis which allows the construction of two similar triangles. In other words, the group recognizes the sum of angles and the product of the norms as the transformation of a triangle into another, similar triangle. We can recognize this transformation by a rotation of factors respecting the placement indicated by the sum of the angles and product of the norms. We can also recognize the semiotic genesis originating from the cognitive level of the MWS as the result of the mediation of our semiotic sign artifact, the icon of ‘Thales’ theorem.</td>
</tr>
<tr>
<td>Semiotic mediation of the sign-artifact</td>
<td>Constructions of the product reflect an influence produced by our sign-artifact. Therefore, there was necessarily a transposition of Descartes’ configuration to the Cartesian plane and thus the recognition of a zero angle in it; recognition explains the correct positioning of the product in cases where the angles of factors are not zero. The construction of similar triangles by C1-I14 could not</td>
</tr>
<tr>
<td>Geometrical meaning of multiplication</td>
<td>simply result from implementation of the properties of multiplication of complex numbers.</td>
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<td></td>
<td>There is evidence of a geometric meaning of the product: the meaning of this mathematical operation is visible in constructions made in response to question 4.b. The product can be interpreted as the transformation of a triangle into a similar triangle by enlargement (or reduction).</td>
</tr>
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</table>

**Conclusion**

New lines of questioning emerge as a result of the diversity of concluding responses and the similarities and differences between the different students’ chosen *paths*. They also demonstrate the difficulty of organizing the different *geneses* of the MWS. We realize that this is quite a complex cognitive activity since there is no direct conversion between one register of representation and another. This leads us to position the students’ activity within a mathematical work space where the meaning of mathematical objects *emerges* as a result of a *cognitive genesis*. This *genesis* assumes the presence of complex semiotic interactions, such as those described by D’Amore and Fandino (2007) in order to describe the difficulties of moving between different representations. For example, the transposition of Descartes’ product to a product operating directly on numbers, represented geometrically, positioned on a plane; this can only be the result of realizing that the activity is based on a *mathematical idea* (Lakoff & Nunez, 1997) that completely departs from the traditional knowledge of the mathematical object in question, i.e. the geometric meanings of multiplication. We are no longer working on the techniques of calculation or proof. Thus, because of the richness of the MWS introduced by the teacher as well as the diversity of students’ personal MWSs, we must highlight the importance of our theoretical combination (MWS (Kuzniak, 2011) and TSM (Bartolini Bussi & Mariotti, 2008)). We based our theoretical framework on a unified conception of cognitive and didactic elements. Several interests informed its development: the social dimension of learning processes; the study of semiotic mediation processes favoring the collaborative construction of a mathematical object; and the construction of meaning of mathematical objects. Through the lens of these theories, we have seen and analyzed students’ ways of looking for the meanings of a mathematical object, dealing with a mathematical sign-artifact between an epistemological and a cognitive level, within a context of social interactions. Finally, the diversity of the proposed and individual MWSs leads us to emphasize the importance of mediation by teachers, which is needed to move towards institutionalization of this multifaceted activity. In addition to mediation related to mathematical sign, there is a second cultural mediation (Radford, 2004) carried out in our case by the teacher. In this cultural mediation, the teacher plays a fundamental role in guiding students through the diversity of Mathematical Work Spaces resulting from a single didactic proposal. Our analysis takes into account new research perspectives on the development of a practical approach to teaching, perspectives considering the relationships between thought and communication and between learning and mediation.
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IREM de Strasbourg.


Appendix
Experimental lesson’s questions and appendix

Travail avec annexe 1
La configuration « Figure A » correspond à ce que Descartes (mathématicien et philosophe, 1596-1650) a considéré comme une représentation géométrique de la multiplication. Nous vous présentons aussi son texte historique la décrivant.

I. Lire le texte de Descartes (annexe 1), puis observer la configuration « Figure A » pour répondre à la question suivante :
   a. Nous proposons AB = 1 cm. BE, est-il bien le produit annoncé par Descartes ? Qu’en pensez-vous ?

   English Translation: The configuration in Figure A corresponds to what Descartes (mathematician and philosopher, 1596 to 1650) considered as a geometric representation of multiplication. We also present his historical text describing this.

I. Read Descartes’ text (Appendix 1) and observe configuration « Figure A » to reply to the following question :
   a. We propose AB = 1cm. Is BE the answer Descartes gives? What do you think ?

Travail avec annexe 2
II. Observer la configuration « Figure B » pour répondre aux questions suivantes :
   a. Nous voudrions construire la représentation géométrique d’un produit ayant les caractéristiques de la multiplication de Descartes :
      - Sachant que BA = 1 cm placer D entre B et A.
      - Construire E pour que BE donne le produit de la multiplication de BD par BC.
   b. Décrire votre construction et expliquer pourquoi elle peut être considérée comme analogue à la multiplication de Descartes.

   English Translation: II. Observe the configuration in « Figure B » to reply to the following questions :
   a. We would like to construct a geometric representation of a response with the characteristics of Descartes’ multiplication :
      - Knowing that BA = 1cm, place D between B and A.
      - Construct E in such a way that BE gives BC as the answer to multiplying BD.
   b. Describe your answer and explain why it can be considered analogous to Descartes’ multiplication.

Travail avec annexe 3
III. Observer la configuration « Figure C » où nous avons repéré sur la droite
numérique certains nombres. Du côté positif nous avons placé le point A d’abscisse +1 et du côté négatif le point D d’abscisse -2. (AC) et (DE) sont parallèles.

a. Projeter orthogonalement les points C et E sur l’axe des abscisses. Pouvez-vous justifier que l’abscisse du point E est le produit des abscisses des points D et C ? Comment ?

b. Si de manière plus générale, l’abscisse du point C est $x_C > 0$ et l’abscisse du point D est $x_D < 0$, décrire la représentation géométrique du point E sur $(BC)$ d’abscisse $x_E = x_C \times x_D$.

**English Translation:** 1. Observe the configuration in “Figure C” where certain numbers on the right are indicated. On the positive side we have placed point A on the X coordinate +1 and on the negative side, point D on the X coordinate -2. (AC) and (DE) are parallel.

a. Place points C and E orthogonally on the X axis. Can you justify that the coordinate of point E is the product of the coordinates of points D and C? How?

b. If in a more general way, the coordinate of point C is $x_C > 0$ and point B is $x_B < 0$, describe the geometric representation of point E on point $(BC)$ of coordinate $x_E = x_C \times x_D$.

**Travail avec annexe 4**

IV. Nous allons étudier une nouvelle configuration, similaire à la configuration précédente, mais celle-ci est située dans le plan complexe avec certains éléments complémentaires.

Soit $(O; \overrightarrow{\text{u}}, \overrightarrow{\text{v}})$, un repère orthonormé direct du plan appelé plan complexe. Nous rappelons qu’un nombre complexe est appelé affixe d’un point M et d’un vecteur $\overrightarrow{OM}$. Utiliser les informations données sur la configuration « Figure D  » pour répondre aux questions suivantes

Pouvez-vous affirmer que dans cette représentation géométrique, la multiplication de $z$ et $z'$ donne toujours le produit $(\text{affixe du point E et du vecteur } BE)$ ? Expliquez votre réponse.

Dans votre réponse à la question précédente, avez-vous établi des liens entre $|BC|$, $|BD|$ et $|BE|$ ? Lesquels ? Et entre $\angle AOC$, $\angle AOD$ (les angles associés aux facteurs) et $\angle AOE$ (l’angle associé au produit) ? Si vous ne les avez pas encore considérés, quels liens établiriez-vous entre ces éléments pour expliquer la représentation géométrique du produit de deux nombres complexes ?

**English Translation:** We are going to study a new configuration, similar to that above, but situated on a complex map with certain complementary elements.

Let $(O; \overrightarrow{u}, \overrightarrow{v})$ be a direct orthonormed location of the map called “complex map.” We remind you that a complex number is affixed to point M and vector $\overrightarrow{OM}$. Use the information given on “Figure D” to reply to the following questions:

- Can you state that within this geometric representation, the multiplication of $z$ plus $z'$ always gives the answer $z''$ (affix of point E and vector $\overrightarrow{BE}$) ? Explain your response.

- In your response to the question above, have you established any relationships between
\[ \| \overrightarrow{BC} \|, \| \overrightarrow{BD} \| \text{ et } \| \overrightarrow{BE} \|. \] What are they? And between \( \overrightarrow{\text{AOE}} \), \( \overrightarrow{\text{AOD}} \) (the angles associated with the factors) et \( \overrightarrow{\text{AOE}} \) (the angle associated with the answer)? If you have not yet considered them, what relationships might you establish between these elements to explain the geometric representation of the product of two complex numbers?

**Travail avec annexe 5**

a. Observez les représentations géométriques de deux nombres complexes dans la configuration « Figure E ».
- En prenant en compte tes réponses dans les questions précédentes construire la représentation géométrique du produit de \( z \) et \( z' \).
- Décrite votre construction.
- Quel lien pouvez-vous établir entre le produit de Descartes et le produit que vous venez de construire?

**English Translation**

a. Observe the geometric representations of two complex numbers in « Figure E ».
- Taking into account your responses to the previous questions, construct the geometric representation of \( z \) plus \( z' \).
- Describe your reasoning.
- What relationship can you establish between Descartes’ answer and your own?

a. **Dernière question** : En vous appuyant sur le travail que vous avez fait dans cette séance et sur vos connaissances antérieures, quelle signification géométrique pouvez-vous donner à la multiplication?

**English Translation**

On the basis of the work you have done in this session and your prior knowledge, what geometric meaning can you give to the multiplication?

**Appendix (annexes) linked to the experimental lesson**

**Annexe 1** (Figure A)

Soit par exemple \( \overrightarrow{AB} \) l’unité et qu’il faille multiplier \( \overrightarrow{BD} \) par \( \overrightarrow{BC} \); je n’ai qu’à joindre les points \( A \) et \( C \), puis tirer la parallèle à \( \overrightarrow{CA} \), et \( \overrightarrow{BE} \) est le produit de cette multiplication (Descartes, 1637).
All for the Success of College Algebra

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Abstract
An extensive support system placed at Texas A&M International University (TAMIU)
for the success of students taking College Algebra course is quintessential, since this
is the first mathematics course taken by those who were admitted or plan to seek admission
to TAMIU. Additional focus has been given as this course is taken by the majority of
students to meet the core mathematics requirement in the state of Texas. Apart from
sections being taught by experienced faculty, the University Learning Center (ULC)
provides walk-in tutoring for students, and each section has been assigned a supplemental
instruction leader who provides additional sessions beyond the classroom teaching, on a
weekly basis. The ALEKS software provides an online homework system. Furthermore, a
marathon review session is planned for the final exam, an opportunity geared towards
procrastinators. This paper elaborates on the extent of the involvement required for the
success of this course.

Keywords: ALEKS, College Algebra, Teaching, Mathematics, ViTAS, SI, Tutoring,
DFW, Online
Introduction
At Texas A&M International University (TAMIU), the offering of college algebra has developed to 13-18 sections and enrollment in Fall 2012 has reached an unprecedented number of 612 students. TAMIU invests enormous amount of assets and resources from funded grants to make this success possible in the course. Thanks to the collective efforts of faculty and administration, TAMIU was able to achieve DFWI (aggregate of all students received grades of D, F, W, and those incompletes) rates at a minimum of 33.00% for the last few years, an endeavor that started back in Fall 2008. Next, some discussions pertaining to success in college courses will be presented to make these efforts relevant.

College students must learn certain requirements that are expected for success in certain social situations that academic context demands. Social situations, however, do not reward students on an intellectual level, contrary to certain institutions’ beliefs. The students’ learning has been dominated by the “grade point average” perspective where high grades in assignments and assessments are considered most important [3]. It is also believed that there must be a solid mathematical knowledge base in order for students to further pursue degrees in Science, Technology, Engineering, and Mathematics (STEM) disciplines. The current US administration has given a great push towards the teaching of STEM disciplines. Therefore, more emphasis should be placed on mathematical teaching through various learning methods. It was strongly believed that instructors should encourage students to pursue mathematical degrees and that instructors should strive for excellence when teaching such a vital subject [7].

Different factors regularly affect student life on campus. A questionnaire concluded that students from different majors described their experiences on campus to be very different environments. Therefore, students from different studies have different needs to better fit their environment and majors [1]. College applications have been declining over the years. College administrators now have to brainstorm creative ways to get students to apply to their colleges. Therefore, the recruitment strategies need to be formulated to ensure enrollment and retention [2]. The relationship between the students’ academic environment and their success has been studied. The researchers came to the conclusion that students’ perceptions greatly influenced their study approach. The students’ perception of the learning environment predicted the learning outcomes of the university as a whole [3]. Educators believe that a stronger student-educator relationship leads to a higher success rate for the student. The greater the student-educator contact, in and outside the classroom, the higher the satisfaction and development of the student. Based on these findings, some universities give incentives to students who pursued the strengthening of their relationship with their educators [4]. A research study was done to see what characteristics both students and faculty thought were necessary for optimizing the learning experience. Out of this, the students thought that the instructor should be
interesting, eloquent, and readily available. The faculty thought that the instructors should be intellectually challenging and motivating with high standards [5]. A study was conducted to target and find the difference between classroom behaviors based on gender. However, the study showed that there is no difference in classroom participation between the two genders. The study also demonstrated that there was no difference in the way the instructors viewed the students of different genders. Rather student participation varied on an individual basis [6].

Data and Analysis

Besides, the opinions collected for the preceding section, it was widely believed that the pace is relatively fast in college mathematics courses. Assignments help students to better understand what is going on in class. The computer assisted projects provide a much needed hands-on approach to these course activities if they are all carefully coordinated to help students grasp the many natural connections among this discipline. The only way students can learn mathematics is by doing it. Homework, in mathematics classes, is assigned to help students understand certain concepts and to help them build certain skills, thereby, understand the process, not the specific problem for the certain topic. These were all part of thinking went into this endeavor.

Table 1 depicts how these efforts have made this investment worthwhile since Fall 2010. Empty cells mean that there were no students enrolled in the categories. Diversity of these offerings has been extended to local high schools thanks to most part from the state of Texas House bill for HB-1 dual credit Initiative enacted in 2006 providing students to earn the equivalent of up to 12 hours of college credit while in their high school. There were instances, where high school students signed in for TAMIU sections to earn concurrence credits. Accordingly, offering of courses has extended to two nearby school districts, United Independent School District (UISD) (two high schools) and Laredo Independent School District (LISD) (one high school) in Spring 2011 and Spring 2012, respectively. Students in these ISDs have been provided supporting instruction (plan) during the days when classes meet, with their in-house school teachers along while they are enrolled in college courses. Early College High School (ECHS), a collaboration of between TAMIU and LISD that began in 2006, admits those high school freshmen who will later enroll in core courses offered by TAMIU as university freshmen. The success of this visionary program was imagined and funded by the Bill and Melinda Gates Foundation. Early College High School provides the dual-enrollment in College Algebra for students at their appropriate levels to meet mathematics core requirement in the curriculum.

Table 1. Diversity of student enrollments from Fall 2010 to Spring 2013

<table>
<thead>
<tr>
<th></th>
<th>TAMIU</th>
<th>UISD</th>
<th>LISD</th>
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<tbody>
<tr>
<td>Classroom</td>
<td>Online</td>
<td>UHS†</td>
<td>JBAHS*</td>
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www.hostos.cuny.edu/departments/math/mtrj
There are several factors to be credited for these accomplishments beyond excellent instructions provided by TAMIU faculty. The University Learning Center (ULC) provides walk-in tutorials and is open for tutoring 11 hours 4 days a week, 7 hours on Thursday and 4 hours on Sunday. Each of the courses has been assigned with a SI (Supplemental Instruction) leader who attends the classes weekly and provides follow-up sessions with supplemental instruction and reviews for exams and homework assistance. ALEKS software became part of instructional materials for students provided assessment tools, and unlimited hours of problem solving experience. In addition, before every final exam, a marathon review session is held usually from 3:00 pm to 8:30 pm covering a set of 60-80 review questions. This Fall the number of students attending this review session reached about 288. A Virtual Teaching Assistant System (ViTAS) is on the other hand, a web based homework grading system provides students, an interactive learning environment outside the class [12]. Using ViTAS, students are able to submit homework online, engage in anonymous homework peer review process and discussion with their peers. Some instructors that taught college algebra courses have been provided with the opportunity of using ViTAS for their students starting back in Summer 2011. This is a Minority Science and Engineering Improvement Program (MSEIP) Project funded by US Department of Education.

What is ALEKS?

Assessment and LEarning in Knowledge Spaces (ALEKS) is a web-based, artificially intelligent assessment and learning system. ALEKS uses adaptive questioning to quickly and accurately determine exactly what a student knows and doesn't know in a course.
ALEKS then instructs the student on the topics he or she is most ready to learn. As a student works through a course, ALEKS periodically reassesses the student to ensure that topics learned are also retained. ALEKS courses are very complete in their topic coverage and it avoids multiple-choice questions. Students who show a high level of mastery of an ALEKS course will be successful in the actual course they are taking. ALEKS also provides the advantages of one-on-one instruction, 24/7, from virtually any web-based computer [8]. ALEKS software is used to provide assignments, homework, and exams, and most importantly, earned hours of practice problems as they advance in the course, as stipulated in the common course syllabus.

For the required curriculum, the first eight chapters from the College Algebra textbook [9] and assessments from the ALEKS 18-week software [8] provided adequate learning guidance for students as well as for instructors. Students are evaluated according to the grading distribution in Figure 1.

![Final Course Grade Calculation](chart)

**Figure 1.** Current final course grade calculation

**ULC**

The primary function of ULC is to develop, implement, and evaluate services specifically designed to enhance learning. ULC provides tutoring in most subjects other than writing and strives to make a positive difference in the lives of the students, staff and community. Another important function of ULC is to serve as a professional resource agency for assistance in the critical area of the University's recruitment and retention efforts. Further, the ULC also assists the University by working directly with local schools and students to help them prepare for University level work. The ULC serves as a place where first year
college students to graduate/professional school students, become more efficient and effective learners [11].

Final Examination must be comprehensive and given on the day specified for all students enrolled in the sections which is a unique feature of this endeavor. The design and structure of these courses have already been well documented [10]. A snapshot of what took place in Fall 2012 is presented to bear a testimony of the extent of these efforts undertaken through several discussions, Table 2, Figures 2a, 2b, 2c, 3, and 4 to follow. Analysis for those who benefitted from SI instructions, marathon review session, and tutoring provided by ULC, and ViTAS assisted assignment submission that is sponsored by the Department of Engineering, Mathematics, and Physics are summarized below:

1) ULC assisted (SI and/or Tutored) vs. non-ULC assisted students
   a. Participation of students in SI and/or tutoring groups
   b. Comparison of students’ GPA in SI and/or tutoring groups
   c. DFWI rates for students in SI and/or tutoring groups

2) Marathon vs. non-marathon review session attended

3) ViTAS vs. non-ViTAS assisted students

![Participation of Students in SI and/or Tutoring Groups](image-url)

Figure 2a. Participation of students in SI and/or tutoring groups in Fall 2012
Figure 2b. Comparison of students’ GPA in SI and/or tutoring groups in Fall 2012

Figure 2c. DFW rates for students in SI and/or tutoring groups in Fall 2012
A comparison of grade distributions for students who attended the marathon review session (held a day before the final exam) and those that did not attend is provided in Figure 3. This does not reflect the extent of this involvement necessarily. However, analysis showed that there is an aggregate gain of 0.53 GPA for those who attended the marathon session in Fall 2012. It can be concluded that the marathon session assisted some students to elevate their grades; thus, achieving the objectives of the session as originally anticipated. In the same token, we can argue that there seem to be a reduction of D’s, F’s and W’s (DFWI’s) for those attended the marathon review session.

Only students in two sections of College Algebra have participated in the ViTAS assisted assignment submission system in Fall 2012. Students in these sections have the opportunity to do their homework at their convenience and submit them online from anywhere in the world. From Figure 4, it can be concluded that this has assisted to improve the aggregate of A, B, and C grades and most importantly, DFWI rates in Fall 2012.
Finally, it is worthy to be noted that this experiment has been paying off. Table 3 summarizes DRWI rates that fell dramatically from Fall 2008 (to Fall 2012).

Table 2. DFWI rates for all Math 1314, College Algebra sections taught since Fall 2008

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<tr>
<td>Total Enrolled</td>
<td>469</td>
<td>308</td>
<td>90</td>
<td>565</td>
<td>418</td>
<td>130</td>
<td>591</td>
<td>497</td>
<td>176</td>
<td>560</td>
<td>480</td>
<td>128</td>
<td>611</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DFWI Count</td>
<td>259</td>
<td>151</td>
<td>42</td>
<td>226</td>
<td>121</td>
<td>40</td>
<td>208</td>
<td>156</td>
<td>34</td>
<td>168</td>
<td>152</td>
<td>17</td>
<td>206</td>
<td></td>
<td></td>
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<tr>
<td>% DFWI</td>
<td>55.22</td>
<td>49.03</td>
<td>46.67</td>
<td>40.00</td>
<td>28.95</td>
<td>30.77</td>
<td>35.19</td>
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<td>19.32</td>
<td>30.00</td>
<td>31.67</td>
<td>13.28</td>
<td>33.72</td>
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+ Summer = both Summer I and II combined

Enrollments in the courses are in an upward trend and success rates are maintained despite emergence of some economic realities for students paying for courses since 2009.

Conclusions
All indications show that this endeavor that began in Fall 2008 has been making a steady progress as initially anticipated. Circumstances could change; however, progress can still be made as the opportunities are available for students to solicit learning. A collective effort of faculty and staff will not leave any stone unturned to make sure successes in these courses are achieved and maintained in the future.
Acknowledgements

The authors wish to thank for funding secured from various grant programs including STEM Recruitment, Retention, and Graduation (STEM-RRG), Minority Science and Engineering Improvement Program (MSEIP). They all have supported various aspects and implementations of college algebra offerings at TAMIU in the past. A superb and excellent support received from Dr. Rex H. Ball, Dr. Juan R. Lira, Dr. Concepcion C. Hickey, Dr. Thomas R. Mitchell, Mr. Mario E. Moreno, Ms. Amelia P. Uribe-Guajardo, and Ms. Aida C. Garza is appreciated. Partial funding received for this project from TAMIU/LCC Title-V grant awarded by the Department of Education allowed to purchase necessary ALEKS software for all students in the courses until Fall 2012. The college algebra instructional team comprised of Patricia D. Proa, Dalia G. Trevino, Teresa N. Nguyen, Miguel San Miguel, Juan M. Gonzalez, Pablo D. Morales, and Joshua J. Edwards deserves authors’ praise for their successful involvement in delivery of the course. The commitment of the ULC staff, tutors, and SI has made this endeavor possible is greatly appreciated too.

Finally, student assistants, Gladys Gonzalez, Navil Lozano, Tony Mendoza, and Vicente O. Ruiz have assisted to improve writings of this article vastly at various stages of preparation.

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[12] ViTAS Main Link: [http://vitas.tamiu.edu/vitasmseip/vitaslogin.aspx](http://vitas.tamiu.edu/vitasmseip/vitaslogin.aspx)
Benefits of Summer Enrichment Workshops for Incoming College Students in STEM Disciplines

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Abstract
It has been discovered that incoming college students need enrichment, support, and guidance in order to overcome the academic challenges ahead of them. Summer enrichment workshops certainly attempt to impact positively these students who plan to pursue programs in Science, Technology, Engineering, and Mathematics (STEM) disciplines. This eventually benefits their ability to complete the programs, timely graduation, and as a result, colleges will also be able to maintain higher retention rates to accommodate growing need of skilled STEM workforce to meet the current demand of the country. This paper summarizes the details of one enrichment workshop series conducted past three summers holding three carefully planned workshops at Texas A&M International University (TAMIU), Laredo, Texas, thanks to the US Department of Education grant to increase recruitment, retention, and graduation of a number of Hispanic and other minority students pursuing degrees in STEM fields at TAMIU.

Keywords: STEM disciplines, enrichment workshops, minority students, participants
Introduction

The grant program, STEM Recruitment, Retention, and Graduation (STEM-RRG), received from the US Department of Education for the academic years 2008-2011 is designed to help Texas A&M International University (TAMIU) establish an innovative recruitment and retention program that increase the number of Hispanic and other minority students pursuing degrees in STEM fields at TAMIU. STEM-RRG consists of several projects that implement a number of activities, including enrichment workshops, scholarships, internships, research experience, mentoring and tutoring, advising and career counseling, experiential training, recruitment of high-potential students, and faculty professional development. To implement these activities, STEM Recruitment and Enrichment Project (STEM-REP) has been established and is expected to accomplish the goals, in particular, to improve the recruitment, enrichment, and preparation of Hispanic and other minority students through participation in summer workshops and a follow-up science and engineering exhibit. The workshop sessions were conducted by discipline-specific instructors, experts, and professionals locally available with a strong emphasis on enhancing critical thinking and problem-solving skills among incoming high school students.

Incoming college students in STEM disciplines have many challenges. These challenges include: adapting to college life, getting used to college courses, coping with professors’ individual teaching styles, and lack of experiences in preparation of laboratory assignments. Accordingly, students face a daunting task in deciding whether to stay focus on college courses or to give up everything altogether amidst of these challenges. These carefully planned two-week long workshops will certainly help retention, in which enrichment is fortified to enhance their ability to succeed in college programs. In a society like the United States, there can be no definite model design for student learning that serves all students and all disciplines. Learning abilities and styles of students are different and unique. These characterize US higher education and remain a source of vitality and strength by itself. Yet all educational institutions and all fields of studies also share this obligation to prepare their graduates to the fullest extent possible for the real-world demands of work, citizenship, and life in a complex and fast-changing world. In this setting, there is great value in a broadly defined educational preparation and objectives of incoming college students who are willing to embark in this important task that provides both a shared obligation, sense of determination, and strong emphasis on effective practices that help students achieve successes in STEM disciplines. To highlight these shared responsibilities, this workshop series has been carefully crafted to impact these young minds as they prepare to meet these college challenges.
Demand for Enrichment Activities

Hispanics comprise a growing segment of population in the United States and a major concern is the shortage of Hispanic students who are pursuing STEM degree programs. The summer enrichment workshops would lead to the recruitment, retention, and graduation of Hispanic students for these programs. Benefit of enrichment activities are essential for incoming college students to succeed in STEM disciplines and degree programs and lead to research and learning potentials.

High school students in the US are expected to have completed a rigorous high-school curriculum, if they receive a diploma, completed a State-designated program or met a similar set of course requirements, or earned at least some achievements in Advanced Placement (AP) courses. Students who complete this coursework either as outlined by their state or under the Education Agency in the state and who seek college admission will be given the opportunities, such as scholarships (Waters, 2006). The pace at which science and mathematics education proceeds in the US has been slow. As such, the U.S. Department of Education requested an appraisal of the scientific knowledge base on human aspects of learning and their application in the current high school education. Research suggests that all of these topics lead to providing all students with inside and outside experience in order to gain the fundamental knowledge base for understanding and implementing necessary changes in student education (Bransford, Brown, and Cocking, 1999). Subsequent enrichment programs can further enhance this knowledge base as they begin to take college courses.

The National Leadership Council for Liberal Education and America’s Promise (LEAP) strongly recommended aligning the essential goals and guiding principles for higher education in today’s dynamic global economy. They also emphasize the importance of providing students with the necessary knowledge and skills but also experience putting those knowledge and skills to practical use. All individuals must master these skills and knowledge in order to be successful and to contribute to today’s global economy with some intervention (Hart, 2004).

Colleges have become a gateway to many opportunities in the scientifically inclined society. The proposed activities must strengthen higher education in the United States, especially the academy’s commitment to inclusion and excellence in advancing innovative solutions to important problems. Working together, with determination, creativity, and a larger sense of shared responsibility, the students in high schools can fulfill the promise of a liberating college education to sustain America’s future through additional enrichment intervention (ABET, 2000). Our nation still has the strongest scientific and technological enterprise and the best research facilities in the world.
However, the recent trend in the country which is occurring in mathematics and science education needs to be rapidly corrected so as not to fall behind of the rest of the world who prospers in this arena (Augustine, 2005).

Students are to be thinking in terms of the professions and more real-world applications in the liberal arts and sciences setting. The other great systemic impediment to educational reform is the ever-changing marketplace and its diversity. When an institution of higher education independently raises its entrance requirements, or holds students to more rigorous graduation standards in science, mathematics, linguistics, or global learning, students remain entirely free to go to another institution that has set less academic standards. For all these reasons, administrators shall work collectively across institutional boundaries as well as within themselves to create conducive environment for the marketplace to flourish and educational excellence is achieved from all. Support and reward for collaborative work must be in place to achieve them (Adelman, 2004).

Extent and Planning of Sessions

It is advocated that small group sessions are favorable for students’ learning and to promote more interaction among the participants. Each of the three workshops was limited to 20 participants to assure extensive interaction between the participants and instructions. In each day, two sessions were conducted from 9:00 am to 4:00 pm MTWRF with a dismissal for lunch from 12:00 pm to 1:00 pm for two weeks in a lecture room equipped with computers totaling 20 sessions of 60 hours of lessons for each workshop under direct supervision of the session leader who is assigned to deliver the enrichment sessions in his/her expertise (see the appendix for session themes). Each session was a complement to other sessions in order to provide a broad range of coverage of subject and discipline matters. In some cases, a split session was held to accommodate an expert in the field. Attendance was taken just before the session started and just before it was over. Alternative sessions were planned to address absences by the participants (1.37%) thus, allowing them to make up hours in emergencies. These workshops have been planned a week apart so that the presence of session leaders can be curtailed to three weeks. This session model that limited the duration of the workshops to four weeks as in Figure 1 was considered important. This allowed the span of any session leader’s engagement only to three weeks unless they chose to conduct two different sessions for the entire workshop series. This format only uses minimum university facilities at any given time. Session topics ranged from mathematics, biology, environmental issues, physics, statistics, business models, geology, computer science, and engineering, to robotics.
The participants contributed to these subject topics by either willing to pursue degree programs in them or choosing minors in these specific fields. Different delivery formats such as lectures, laboratory experiments, library sessions, and planetarium shows were used to accommodate the sessions in groups and individually in an actual college setting. The participants were very curious to learn how to interact with college professors as many of the sessions have been conducted by them. They had access to the campus cafeteria, student center, gymnasium, game room (pool tables etc.), recreational center (exercise equipment, table tennis etc.), bookstore, and library. The lunch break was certainly a valuable experience for many as they completed one session and got prepared for another one. This paved the way to promote collegiality among participants and make new friends among them. Some took the opportunity to explore the University facilities during this time. The last day of activities consisted of presentation of participants’ experience (7-10 minutes of each either oral or PowerPoint self-presentation by the individual or group participants), completion of workshop feedback form by the participants, award of certificates, group photo, and concluding remarks by the Program Director, Dr. Rafic A. Bachnak during the afternoon session. At the end of each presentation, there was a time for Q&A and comments relevant to their talks. Some participants took the opportunity to inquire the STEM degree programs and relevant coursework offered by TAMIU.

One feature of the workshops was to promote presentation skills among participants. This was certainly achieved, as was evident from their brief final day presentations describing their experience during the two-week period. Some chose to have PowerPoint presentations and others used video presentations to deliver their extents of experience. There were many group presentations as well.
Recruitment and Selection of Participants

About one-hundred fifty application packages were distributed to area high school counselors through the University’s the Office of High School Recruitment. Each application package contained of a program flyer, program benefits, schedule of sessions (see, appendix 1), and an application each year. Another 100 application packages were sent to area high school through students working for the programs. Some freshman students in the University were encouraged to recruit seniors and juniors at local area high schools. About 100 applications have been received. Selections were done thereafter. All twelfth graders wanted to pursue STEM disciplines were given a priority. Eleventh graders were selected from the applications received who indicated strong interests to pursue programs leading to STEM disciplines at TAMIU. Criteria for selection are the strength of the students’ essays and interests in STEM disciplines. Their high school GPA was certainly a factor in the decision-making process. Selected students were communicated using e-mails and phone calls. Some instances, students were contacted to clarify items in the applications including e-mail addresses and missing information, if any, and to provide another preference for a workshop. Final acceptance letters were issued to the selected students close to the actual workshop.

Analyses and Extent of Achievements

Table 1 provides the breakdown of the 51 workshop participants attended the workshop series held in summer of 2009 on the basis of the workshop they attended, the grade level, and gender. Most of them are either incoming college freshman or juniors at area high schools taking concurrent courses at TAMIU. For the remaining two workshop series, 67 students attended in the summer of 2010 and 39 attended in 2011 totaling 157 high school students.

<table>
<thead>
<tr>
<th>Workshop 1: June 1-12, 2009</th>
<th>Workshop 2: June 8-19, 2009</th>
<th>Workshop 3: June 15-26, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>Female</td>
<td>Male</td>
</tr>
<tr>
<td>12th Graders</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>11th Graders</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>7</td>
</tr>
</tbody>
</table>

All participants have been requested to comment on their experience by answering a set of eleven questions in a feedback form (appendix 2 provides the form used in 2009) at the end of their last day of the respective workshops. Figures 2a-d depicts the aggregate of these responses gathered using the feedback forms from the workshops 1, 2, and 3 for questions 1-7 for the years, 2009, 2010, and 2011, respectively. In addition, all participants from the workshops 1, 2, and 3 indicated (questionnaire 9) that they would overwhelm recommend this workshop to a friend.
Figure 2a. Aggregates of the responses for questions 1-7 received from the participants who attended workshops 1, 2, 3, overall for the workshops held in 2011

Figure 2b. Achievement of STEM experience following workshops held in 2011
For the workshop held in 2011, an additional set of questionnaires has been included as advised by the project external evaluator. Figure 2b depicts the aggregate results to show that the students have achieved wide range of STEM experience following the workshops.

**Figure 2c.** Aggregates of the responses for questions 1-7 received from the participants who attended workshops 1, 2, 3, overall for the workshops held in 2010
**Figure 2d.** Aggregates of the responses for questions 1-7 received from the participants who attended workshops 1, 2, 3, overall for the workshops held in 2009

For the three open-ended questions (8, 10, and 11) in the feedback form, Table 2 summarizes the responses received in 2009 as to how the majority of them responded using top six responses received (two responses for each workshop). Similar responses were received in 2010 and 2011. These responses will be certainly taken into consideration when planning similar workshops in the future.

**Table 2.** Top six responses received for questions 8, 10, and 11, respectively

<table>
<thead>
<tr>
<th>Workshop 1:</th>
<th>Workshop 2:</th>
<th>Workshop 3:</th>
</tr>
</thead>
<tbody>
<tr>
<td>June 1-12, 2009</td>
<td>June 8-19, 2009</td>
<td>June 15-26, 2009</td>
</tr>
</tbody>
</table>

8. In what ways could this workshop have been improved to better suit your needs?
- Provide lunch
- More hands-on activities
- Shorter hours
- Three-week workshop
- Extended lunch break
- More labs or projects

10. Please comment on how sessions helped you grasp mathematical and science concepts
- Helped to remember old materials
- Students got to understand and learn new concepts
- Uses of concepts in multiple ways
- Students got to understand and learn new concepts
- Helped to remember old materials
- Good lectures by experienced professors

11. Please comment on your overall experience during these workshop sessions
- Enjoyed the program
- Excellent educational experience
- Enjoyed topics of the program
- Great experience
- Very interesting
- Learned new materials

**Participants’ Experience in their Own Words**

All participants echoed their experiences and benefits received from this two-week workshop during the final session just before the awarding of certificates and group photos. Some have remarked about particular sessions, session leaders, and an activity stemmed from a session or two. The following is the summary of these brief presentations as our student assistants were able to witness, gather, and take notes. Some remarks were paraphrased and corrected for inclusion in the paper.

The workshops allowed them to meet new people from the area who share the same passion and desire to succeed in mathematics and science. Students were made more aware of the role of mathematics in the world today; “mathematicians are becoming the global elite” as one said. In relation to the role of mathematics in scientific fields, one student remarked that he learned “mathematics is the language of science.” A majority of students acknowledged that, not having taken a mathematics course within the past semester, the sessions which concentrated on the practice of algebra helped to greatly
“refresh their mind.” The session dealing with the application of statistics in the world of criminal justice helped students make a connection between mathematics and social behavior. Students learned about the pressing environmental issues right here in Laredo, most notably the “plastic bag problem.” Along with a trip to the planetarium, students enjoyed learning about the study of astronomy and the methods of mathematics that were used by scholars as early as Galileo to determine the dimensions of the universe. Coming back to Earth, many students made mention of the hands-on-lab session focusing on rocks and minerals in which they were challenged to differentiate and identify rocks from an array of samples. Most students admitted that although they hadn’t really considered geology before, the lab was fun and interesting. The overall message from the students was, “we learned how to apply mathematics in real life applications” and “I enjoyed and benefited from the experience.” Suggestions for more physical activities, shorter session periods, supply of food, and more advanced mathematics material were among those mentioned by some participants.

Probably the most appreciated session was that of algebraic principles such as polynomials and trigonometry. The Lego Robotics session gave students the chance to design their own unique robots and taught them about the use of a variety of different sensors in the field of engineering. The students all enjoyed this hands-on session. Aside from the robotics session, the lessons held using the AutoCAD and Geometer’s Sketchpad software helped aspiring engineers and architects gain insight into the inner workings of the profession. The session dealing with the practice and applications of mathematics seemed to really help the students to refresh their skills and even learn new ways to solve quadratic equations and use matrices. Participants really enjoyed the session on genetics and learned about the importance of gene study and how unique we all are. They mentioned on several occasions that they enjoyed session leader’s style of teaching and felt confident in his ability to answer their questions; students planning to major in biology were encouraged by these remarks. Regardless of whichever academic path these students may have had in mind when they began the workshop, they all left with a greater understanding and appreciation for mathematics; “mathematics is the leading principle of life which brings people together as a unit.” Some students even decided to switch their major to mathematics because of the positive experiences provided by the sessions and professors. Students credited the effectiveness of the sessions to the “high quality of the professors.”

Students learned about the issues involving water treatment and processing and were inspired to take a more active role in community efforts to make Laredo a cleaner place. They expressed deep gratitude for the opportunity to learn from such knowledgeable and interesting professors and seemed to have retained most of what they learned during the two weeks. Students made group oral and video presentations; one of which included a jeopardy style question and answer section at the end. The enthusiasm with which these
participants spoke and presented is a strong indicator that these workshops achieved their goals to inspire and motivate young minds to excel in the fields of STEM.

Conclusions
This workshop series was a great success as a total of 157 high school students were able to benefit from three two-week summer workshops consecutively held from 2009 to 2011. The attendance has been enforced to its fullest extent. As a result, the program administrators were able to achieve a high rate of attendance throughout all sessions of three workshops by the participants. The participants echoed the positive impact of the workshop on them to make their college life successful. The feedback received from the workshops 1, 2, and 3 for each year of the program will be used in planning the similar future workshops of this type. This paper provides a model of summer enrichment workshops that can be used to prepare incoming college students to succeed in STEM education. This also provides development of new enrichment activities to increase preparedness of Hispanic students for STEM education together with adequate added benefits to incoming Hispanic college students to successfully complete STEM degrees on time.

Acknowledgements
The STEM-REP project is partially supported by a STEM Recruitment, Retention, and Graduation (STEM RRG) project funded by the U.S. Department of Education (Award # P031C080083). More information about this grant is found at http://www.tamiu.edu/~rbachnak/STEMRRG/index3.html. The support received from the entire team of session leaders was enormous. The Office of Information Technology and the Office of Recruitment of the University deserve our appreciation. The student assistants have taken care of the day to day activities of the workshops while assisting the authors. Their patience was acknowledged. Finally, large thanks goes to Ms. Juanita Villarreal, the department secretary who assisted in purchasing and providing items needed for a successful completion of the workshop and arranging quick payments for session leaders of the STEM-REP workshops.

REFERENCES


Appendix 1. STEM Recruitment and Enrichment Project (STEM-REP) Schedule for Speakers and Session Themes Schedule for the 2008-2009 Academic Year

<table>
<thead>
<tr>
<th>Week 1</th>
<th>Workshop 1: June 1 – 12, 2009</th>
<th>Workshop 2: June 8 – 19, 2009</th>
<th>Workshop 3: June 15 – 26, 2009</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>9:00 am – 12:00 pm</td>
<td>9:00 am – 12:00 pm</td>
<td>9:00 am – 12:00 pm</td>
</tr>
<tr>
<td>M</td>
<td>Welcome &amp; Introduction</td>
<td>Introduction to computer hardware</td>
<td>A primer on genetics</td>
</tr>
<tr>
<td></td>
<td>Dr. David L. Beck</td>
<td>Dr. Roberto R Heredia</td>
<td>Dr. David L. Beck</td>
</tr>
<tr>
<td>T</td>
<td>Robotics with legos</td>
<td>Local Environmental issues</td>
<td>Robotics with legos</td>
</tr>
<tr>
<td></td>
<td>Dr. Fethi Belkhouch</td>
<td>Mr. Gerardo Pinzon</td>
<td>Dr. Fethi Belkhouch</td>
</tr>
<tr>
<td>W</td>
<td>Decision support in operations management using Microsoft Excel</td>
<td>Practical uses for matrices</td>
<td>Practical uses for matrices</td>
</tr>
<tr>
<td></td>
<td>Dr. Balaji Janamanchi</td>
<td>Prof. Miguel San Miguel</td>
<td>Prof. Miguel San Miguel</td>
</tr>
<tr>
<td>R</td>
<td>Applying statistical techniques to real-world problems</td>
<td>Several methods for solving quadratic equations</td>
<td>Several methods for solving quadratic equations</td>
</tr>
<tr>
<td></td>
<td>Prof. Dae-Hoon Kwak</td>
<td>Prof. Alma V. Jasso</td>
<td>Prof. Alma V. Jasso</td>
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<tr>
<td>F</td>
<td>Exploring the universe</td>
<td>Exploring the universe</td>
<td>Exploring the universe</td>
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<td>Dr. Juan H Hinojosa</td>
<td>Dr. Juan H Hinojosa</td>
<td>Dr. Juan H Hinojosa</td>
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<tr>
<td>M</td>
<td>A deeper understanding of mathematics through mathematical puzzles</td>
<td>A deeper understanding of mathematics through mathematical puzzles</td>
<td>A deeper understanding of mathematics through mathematical puzzles</td>
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<td></td>
<td>Mr. Timothy E. Bogue</td>
<td>Mr. Timothy E. Bogue</td>
<td>Mr. Timothy E. Bogue</td>
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<tr>
<td>T</td>
<td>The role of mathematics and science in the business world</td>
<td>The role of mathematics and science in the business world</td>
<td>The role of mathematics and science in the business world</td>
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<tr>
<td></td>
<td>Mr. Mario E. Moreno</td>
<td>Mr. Mario E. Moreno</td>
<td>Mr. Mario E. Moreno</td>
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</table>

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| W | Working with geometer's sketchpad  
Mr. Pablo David Morales | The rock cycle: how earth materials change with time  
Dr. Marvin E Bennett | Working with geometer's sketchpad  
Mr. Pablo David Morales | The rock cycle: how earth materials change with time  
Dr. Marvin E Bennett | Working with geometer's sketchpad  
Mr. Pablo David Morales | The rock cycle: how earth materials change with time  
Dr. Marvin E Bennett |
|---|---|---|---|---|---|---|
| R | Polynomial and rational functions  
Prof. Alma V. Jasso | Exploring new worlds: the search for extrasolar planets (in Planetarium)  
Mr. Gerardo A. Perez | Polynomial and rational functions  
Prof. Alma V. Jasso | Exploring new worlds: the search for extrasolar planets (in Planetarium)  
Mr. Gerardo A. Perez | Polynomial and rational functions  
Prof. Alma V. Jasso | Exploring new worlds: the search for extrasolar planets (in Planetarium)  
Mr. Gerardo A. Perez |
| F | Contours and gradients  
Profs. Joe McCary | Presentation of participants’ experience & award of certificates  
Profs. Joe McCary | Contours and gradients  
Profs. Joe McCary | Presentation of participants’ experience & award of certificates  
Profs. Joe McCary | Applying statistical techniques to real-world problems  
Prof. Dae-Hoon Kwak | Presentation of participants’ experience & award of certificates  
Prof. Dae-Hoon Kwak |

For information of this program, visit: [http://www.tamiu.edu/~rbachnak/STEMRRG/Files/REP%20Mathematics%20Enrichment%20Flier.pdf](http://www.tamiu.edu/~rbachnak/STEMRRG/Files/REP%20Mathematics%20Enrichment%20Flier.pdf)

**Appendix 2. Workshop Feedback Form: STEM Recruitment and Enrichment Project (STEM-REP), Summer 2008-2009**

**Review Guidelines**

Please take a moment to complete this feedback form. Your comments will assist us in improving our future workshops and seminars.

* This information is confidential and will only be used by the project personnel *

**Participant Information**

| Name (Optional): | Error! Not a valid bookmark self-reference. | Date: |

**Evaluation**

<p>| Scale: | 5 - strongly agree; 4 – agree; 3 – neutral; 2 – disagree; 1 – strongly disagree |
|---|---|---|---|---|
| 1. Information and Communication before the workshop was timely and accurate. |  |  |  |  | Error! Not a valid bookmark self-reference. |
| 2. The workshop was scheduled at a suitable time. |  |  |  |  | Error! Not a valid bookmark self-reference. |
| 3. The workshop facilities and location were appropriate and satisfactory. |  |  |  |  | Error! Not a valid bookmark self-reference. |
| 4. The workshop material was presented in a clear and organized manner. |  |  |  |  | Error! Not a valid bookmark self-reference. |</p>
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<tr>
<td>5. The staff responded to questions in an informative, appropriate and satisfactory manner.</td>
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<tr>
<td>6. Handouts (if provided) were clear and useful.</td>
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<tr>
<td>7. Overall, the session was valuable and added to my understanding of mathematics.</td>
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<tr>
<td>8. In what ways could this workshop have been improved to better suit your needs?</td>
<td></td>
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<td></td>
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<tr>
<td>9. Would you recommend this workshop to a friend?</td>
<td>Yes</td>
<td></td>
<td>No</td>
<td></td>
<td>Error! Not a valid bookmark self-reference.</td>
</tr>
<tr>
<td>10. Please comment on how sessions helped you grasp mathematical and science concepts</td>
<td></td>
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<td></td>
<td></td>
<td>Error! Not a valid bookmark self-reference.</td>
</tr>
<tr>
<td>11. Please comment on your overall experience during these sessions</td>
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BELIEFS ABOUT SOCIAL JUSTICE AMONG ELEMENTARY MATHEMATICS TEACHERS

Brian R. Evans
Pace University
New York, United States

Abstract

The purpose of this study was to measure teacher beliefs about social justice over the course of an elementary mathematics teaching methods course. The participants in the study came from three unique groups of in-service and preservice teachers in a master’s degrees program at a medium-size university in New York: New York City Teaching Fellows (NYCTF), Teacher Education Assessment and Management (TEAM) program, and traditional preservice teacher preparation program. Findings revealed that while there were no differences in beliefs over the course of the semester, NYCTF teachers had more positive beliefs about social justice than did TEAM teachers. Teachers felt most positively about incorporating diverse cultures and experiences into classroom lessons and discussions; self-examination of attitudes and beliefs about race, class, gender, disabilities, and sexual orientation; and teaching students to think critically about government positions and actions.
Mathematics has been called a “gatekeeper” for quality college education and higher paying careers (D’Ambrosio, 2012; Gau Bartell, 2012; Gonzales, 2012; Koestler, 2012; Leonard, 2008; Martin, 2002; Moses & Cobb, 2001; Stinson, 2004), which means mathematical literacy is a necessary, but not sufficient, condition for later success. Gonzales (2012) said that “knowledge of mathematics is often a prerequisite for full and successful participation in society” (p. 128). There is a direct relationship between success in mathematics and success later in life, and this is even more important for students of lower socioeconomic status and commonly underrepresented groups such as African American, Latina/o, and female students (Gonzales, 2009; Thomson & Hillman, 2010). The careers that offer the best working conditions and highest salaries generally require strong mathematics background such as mathematician, actuary, statistician, software engineer, and computer systems analysis (Needleman, 2009). D’Ambrosio (2012) said, “The institutions of modern civilization—mainly economics, politics, management, and social order—are rooted in mathematics” (p. 209). However, all careers require quantitative fluency to varying extents. Not only will proficiency in mathematics be necessary to help students gain acceptance into quality colleges with the financial assistance needed for them to attend, and obtain positions in desirable careers, but proficiency in mathematics will be necessary to make informed decisions required of all citizens including important economic, social, and political decisions (Gutstein & Peterson, 2005). Knowledge of mathematics is an important condition for financial literacy, and is necessary for citizen decision-making in a healthy democracy. Mathematics teachers, especially at the elementary level, must help all students succeed in mathematics given the subject’s importance for later life satisfaction. Particularly important is the need for teachers of underserved students to have the necessary attitudes, beliefs, and dispositions toward social justice in teaching mathematics to these students.

Caution should be taken regarding the type of mathematics students need for future success given the ever accelerating changes technology and globalization have facilitated in the modern job market. D’Ambrosio (2012) said, “To prepare children to be proficient in obsolete mathematics is to prepare them to the anguish of being marginal in the future, because they will possess outdated knowledge” (p. 206). Certain types of mathematical knowledge transcend the effects of technology and globalization, namely good problem solving and critical thinking abilities. This means the mathematics envisioned for marginalized students is the type of mathematics that scholars and national organizations have promoted as best practice in mathematics (National Council of Supervisors of Mathematics [NCSM], 1978; National Council of Teachers of Mathematics [NCTM]; 2000; Schoenfeld, 1985). NCSM (1978) said problem solving is the principal reason for studying mathematics. Additionally, NCTM (2000) said, “Problem solving is not only a goal of learning mathematics but also a major means of doing so” (2000, p. 52). The important aspect to this issue is to teach this type of mathematics in a culturally responsive manner.
Social Justice in Mathematics

Stinson and Wager (2012) said that teaching for social justice in mathematics is “rooted, in part, in the belief that all children should have access to rich, rigorous mathematics that offers opportunities and self-empowerment for them to understand and use mathematics in their world” (p. 10). To facilitate this, teachers for social justice in mathematics need to be introspective toward their own identities as agents, as both individuals and teachers, for social change (Gonzales, 2009; Leonard, Brooks, Barnes-Johnson, & Berry, 2010). Gau Bartell (2012) said teachers are generally unprepared “to teach mathematics to an increasingly racially, ethnically, linguistically, and socioeconomically diverse student population with which they often have had limited previous interactions” (p. 113). Mathematics teachers must be given the opportunities to directly reflect upon their own conceptions of social justice and learn how to teach from a social justice perspective, which can be conducted through professional development and work with after school programs (Gonzales, 2009; Leonard & Evans, 2008). Leonard and Evans (2012) said, “An explicit focus on teachers’ beliefs and expectations [about social justice] should be a component of (mathematics) teacher education” (p. 101).

There are two approaches to social justice considered in this study. First, a social justice perspective supports students from traditionally underserved groups to receive the quality education they deserve. Historically education in the United States developed to create a skilled and educated workforce for industry, which means that specific groups were prepared for specific roles (Gutstein, 2012; Stinson & Wager, 2012). Further, “different schooling experiences... support class division... [and] also produce and reproduce these unjust divisions through the differing... curricular that are made available” (p. 7). Culturally responsive pedagogy (CRP), which considers student backgrounds for delivering engaging and meaningful mathematics lessons based upon student needs (Gay, 2000; Ladson-Billings, 1994; Leonard, 2008), is an important component to teaching for social justice. CRP is in opposition to a “color-blind” approach and not only helps teachers to carefully consider, respect, and appreciate the various cultural backgrounds and experiences students bring to the classroom, but also helps caring teachers guide students in their developmental and academic success. Student cultural and ethnic identities should be carefully considered in order for teachers to differentiate their instruction to the students’ learning needs (Gay, 2000; Leonard, 2008; Ladson-Billings, 1994; Martin, 2007).

Second, social justice is considered in terms of the mathematical content presented to school students and how the students can use this knowledge. Stinson and Wager (2012) said social justice mathematics helps to “prepare students to take action and use mathematics for social change” (p. 10). The teaching of any subject is inherently political and not neutral as many might suppose. Frankenstein (2012) said, “Reflecting on knowledge means that we understand the non-neutrality of all knowledge, and the connections between knowledge and power” (p. 58), and concluded that knowledge is not neutral. Further, Gutstein and Peterson (2005) said, “Simply put, teaching math in a neutral manner is not possible. No math teaching—no teaching of any kind, for that
matter—is actually ‘neutral,’ although some teachers may be unaware of this” (p. 6). For example, a mathematics problem might require students to perform basic operations on a dilemma of how much money they have and how much fast food they can purchase. However, this type of problem promotes unhealthy eating, environmental degradation instigated by the fast food industry, and animal cruelty through factory farming. A better problem would be one in which students quantitatively examine the nutritional content of fast food or the statistics on the environmental impact and animal cruelty involved in factory farming. A problem that is similar to the one in which children determine the amount of money needed for fast food purchases would require students to perform operations on funds required for purchasing healthy food from the local market, such as whole grains, legumes, vegetables, and fruits. While there may be objections that having children determine the money needed to buy fast food entertains their interests, since many children enjoy eating fast food, a similar argument could be made that many teenagers enjoy smoking. However, teachers would never consider giving students a mathematics problem that involved purchasing cigarettes. Gutstein and Peterson (2005) argued that one cannot be neutral in education because by not contextualizing mathematics in a political setting one is also making a political statement that certain issues are not important. Gutstein and Peterson (2005) said teacher choices teach students three things:

1. They suggest that politics are not relevant to everyday situations.
2. They cast mathematics as having no role in understanding social injustice and power imbalances.
3. They provide students with no experience using math to make sense of, and to change, unjust situations. (p. 6)

An example of empowering student decision making through high school statistics involves determining if a company employed racist policies in its selection of a committee. The problem, adapted from Larson and Farber (2006), states:

You have been selected for jury duty. You decide to serve your civic duty and arrive at the court house. The trial involves a company being accused of racist procedure. This company has 200 employees and it claims that it chose a committee of 15 at random to represent employee retirement issues. When the committee was formed, none of the 56 employees of color were selected.

1. Find the number of ways 15 employees can be chosen from 200.
2. Find the number of ways 15 employees can be chosen from 144 White employees.
3. If the committee was chosen at random (without bias), what is the probability that it contained no employees of color?
4. Does your answer in part 3 indicate that the committee selection was biased?

This problem requires students to determine that the probability of selecting no employees of color for the committee is about 0.005832, which is very low and very unlikely to have occurred by chance. Without mathematical analysis, it would not be
possible to determine that there was a strong likelihood that discrimination had occurred in the selection of the committee.

**Theoretical Framework**

This study is grounded in critical race theory (CRT) in education (e.g., Ladson-Billings & Tate, 1995), which examines race and racism as it applies to education. CRT acknowledges that racism is pervasive throughout society, which means in an educational context racism affects not only children’s learning, but also all aspects of the social and academic realities of schooling. Through working within a CRT framework educators have the objective, both in educational and broader contexts, of eliminating racial oppression as a subset of the goal of eliminating all types of oppression (Dixon & Rousseau, 2005; Matsuda, Lawrence, Delgado, & Crenshaw, 1993). Cultural responsiveness, a social justice orientation, and fostering of trust and care in the classroom are important components for educating students who have been traditionally underrepresented in mathematics related fields (Haberman, 1991; Ladson-Billings, 1994; Leonard, Napp, & Adeleke, 2009). While strong content knowledge is important for effective teaching, these variables are equally important in their impact on learning for underrepresented students (Martin, 2007).

This study is also grounded in Freire’s (1970/2000) concept of *Conscientização*, or critical consciousness, which allows the individual to critically perceive injustice and provide an intellectual means for opposing injustice. Teachers need to be given the opportunity to critically approach injustice in society in general, and in the schools in particular, as an important process in assisting them to critically reflect upon institutional and personal teaching practices.

**Purpose of the Study and Background on Participants**

The purpose of this study was to measure teacher beliefs about social justice over the course of an elementary mathematics teaching methods course at a university that emphasizes social justice for teachers in its conceptual framework. The participants in the study came from three unique groups of in-service and preservice teachers in master’s degrees programs at a medium-size university in New York: New York City Teaching Fellows (NYCTF), Teacher Education Assessment and Management (TEAM) program, and traditional preservice teacher preparation program. The two-year graduate program for all three groups was designed to prepare teachers to teach in urban schools in New York with certification in elementary and special education.

The NYCTF program is an alternative certification program developed in 2000 in conjunction with the New Teacher Project and the New York City Department of Education (Boyd, Lankford, Loeb, Rockoff, & Wyckoff, 2007). NYCTF had the goal of bringing professionals from other careers to fill large teacher shortages in New York public schools. When the program began there was a prediction of a 7000 teacher shortage and fears that there could be up to 25,000 teacher vacancies in the early first decade of the 21st century (Stein, 2002). NYCTF teachers begin their program in June when they are immersed in coursework at their partnering universities in New York. In September they become the teachers of record in their classrooms while continuing their
graduate work in education in a master’s degree program. NYCTF teachers receive a Transitional B teaching license in New York State, which is valid for three years provided they remain with the program and complete program requirements. After this successful completion of this commitment they are eligible to apply for initial certification.

The TEAM program is a collaboration between the TEAM organization and the partnering university. TEAM is an organization that facilitates partnerships with universities on behalf of student members who then receive tuition discount (TEAM, 2012). The university partnership began in 2009 and had its first cohort begin the program in 2010. Cohorts consist of 12 to 20 Orthodox Jewish teachers separated by sex in the classroom due for religious purposes but in this study all TEAM participants were women. TEAM participants enrolled in the program to prepare for certification to teach in Yeshiva and Hebrew Academies. Several of the participants were already teaching in Yeshiva and Hebrew Academies during this study, while most of the participants were preparing to become teachers, but not currently teaching.

Traditional preservice teachers were enrolled in the university’s graduate program, which required extensive fieldwork. Participants in the program were required to have 10 hours of fieldwork for each three credit class in which they were enrolled. Much of the work done in the classes was related to the fieldwork experience including lesson and unit planning, as well as reflection on the teaching experience. Participants in the program were encouraged to incorporate the theory and teaching techniques they were learning in their graduate program into classroom practice.

Research Questions

1. Were there differences in beliefs about social justice over the course of a semester in a reformed-based mathematics methods course?
2. Were there differences in beliefs about social justice between the NYCTF, TEAM, and traditionally prepared teachers?
3. What were teacher beliefs about social justice in the classroom?

Methodology

This study used a quantitative methodology and the sample consisted of 115 preservice and new in-service teachers. All NYCTF teachers were in-service teachers, and TEAM and traditional teachers were preservice teachers, with several TEAM participants teaching in Yeshiva and Hebrew Academies. There were 84 NYCTF teachers, 16 TEAM teachers, and 15 traditional teachers. Participants were enrolled in an inquiry- and reformed-based elementary mathematics methods course that involved both pedagogical and content instruction and was aligned with the NCTM Principles and Standards for School Mathematics (2000).

Teachers were given the Learning to Teach for Social Justice Scale (LTSJ) at the beginning and end of the semester, which was developed by Enterline, Cochran-Smith, Ludlow, and Mitescu (2008) and Ludlow, Enterline, and Cochran-Smith (2008), and measured participants’ beliefs about teaching from a social justice perspective. The LTSJ is a 12-item 5-point Likert scale instrument that solicits participant beliefs about social
justice in the classroom based upon diversity issues such as race, culture, language, gender, disability, and sexual orientation.

**Results**

Paired-samples *t*-test was conducted to answer research question one in order to determine differences in the LTSJ scores over the course of the semester. No statistically significant differences were found.

One-way ANOVA was conducted to answer research question two in order to determine differences in LTSJ scores between NYCTF, TEAM, and traditional teachers. A statistically significant difference was found at the 0.05 level for pre- and post-test LTSJ scores with $F(2, 112) = 3.592, p = 0.031$, $\eta^2 = 0.06$ and $F(2, 112) = 5.247, p = 0.007$, $\eta^2 = 0.09$, respectively. A post hoc Tukey HSD test was conducted to determine exactly where the means differed among the programs (see Table 1). On the pretest it was found NYCTF teachers ($M = 3.94, SD = 0.459$) had more positive dispositions toward social justice than did TEAM teachers ($M = 3.61, SD = 0.433$) with $p = 0.023$. On the posttest it was also found NYCTF teachers ($M = 3.94, SD = 0.529$) had more positive dispositions toward social justice than did TEAM teachers ($M = 3.51, SD = 0.382$) with $p = 0.006$. The effect sizes for both pretest and posttest were in the small to medium range. There were no other statistically significant differences.

Table 1

<table>
<thead>
<tr>
<th>Learning to Teach for Social Justice Scale (LTSJ) Scores</th>
<th>Pretest Mean (SD)</th>
<th>Posttest Mean (SD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NYCTF</td>
<td>3.94* (0.459)</td>
<td>3.94** (0.529)</td>
</tr>
<tr>
<td>TEAM</td>
<td>3.61* (0.433)</td>
<td>3.51** (0.382)</td>
</tr>
<tr>
<td>Traditional</td>
<td>3.87 (0.443)</td>
<td>3.76 (0.399)</td>
</tr>
</tbody>
</table>

*Note. N = 115.*  
* $p < 0.05$. ** $p < 0.01$.

Descriptive statistics were used to answer research question three (see Table 2). Results indicated teachers felt most positively about incorporating diverse cultures and experiences into classroom lessons and discussions; self-examination of attitudes and beliefs about race, class, gender, disabilities, and sexual orientation; and teaching students to think critically about government positions and actions. Teachers felt most positively about negative attitudes such as preparing students for lives they are most likely to lead; student success in school being dependent on how hard they work; and the teachers’ jobs as not being agents of societal change. These final three items reflect negative attitudes toward social justice.
### Table 2
**Survey Results for Beliefs about Social Justice**

<table>
<thead>
<tr>
<th>Learning to Teach for Social Justice Scale (LTSJ)</th>
<th>Pretest Mean</th>
<th>Posttest Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. An important part of learning to be a teacher is examining one’s own attitude and beliefs about race, class, gender, disabilities, and sexual orientation.</td>
<td>4.27</td>
<td>4.28</td>
</tr>
<tr>
<td>2. Issues related to racism and inequity should be openly discussed in the classroom.</td>
<td>4.08</td>
<td>3.93</td>
</tr>
<tr>
<td>3. For the most part, covering multicultural topics is only relevant to certain subject areas, such as social studies and literature.</td>
<td>4.00</td>
<td>3.99</td>
</tr>
<tr>
<td>4. Good teaching incorporates diverse cultures and experiences into classroom lessons and discussions.</td>
<td>4.60</td>
<td>4.50</td>
</tr>
<tr>
<td>5. The most important goal in working with immigrant children and English language learners is that they assimilate into American society.</td>
<td>3.75</td>
<td>3.74</td>
</tr>
<tr>
<td>6. It’s reasonable for teachers to have lower classroom expectations for students who don’t speak English as their first language.</td>
<td>4.12</td>
<td>4.10</td>
</tr>
<tr>
<td>7. Part of the responsibilities of the teacher is to challenge school arrangements that maintain societal inequities.</td>
<td>3.90</td>
<td>4.05</td>
</tr>
<tr>
<td>8. Teachers should teach students to think critically about government positions and actions.</td>
<td>4.17</td>
<td>4.18</td>
</tr>
<tr>
<td>9. Economically disadvantaged students have more to gain in schools because they bring less to the classroom.</td>
<td>3.99</td>
<td>3.93</td>
</tr>
<tr>
<td>10. Although teachers have to appreciate diversity, it’s not their job to change society.</td>
<td>3.70</td>
<td>3.51</td>
</tr>
<tr>
<td>11. Whether students succeed in school depends primarily on how hard they work.</td>
<td>3.15</td>
<td>3.11</td>
</tr>
<tr>
<td>12. Realistically, the job of a teacher is to prepare students for the lives they are likely to lead.</td>
<td>2.90</td>
<td>2.96</td>
</tr>
</tbody>
</table>

*Note.* $N = 115$.  
Items are from Enterline et al. (2008) and Ludlow et al. (2008). Negative items were reversed scored so that high scores still represented positive attitudes (items 3, 5, 6, 9, 10, 11, and 12).
Discussion

It was found that while there were no differences in beliefs about teaching for social justice over the course of the semester, teachers from the NYCTF program had more positive beliefs about social justice than did teachers from the TEAM program. This result was not surprising given that teachers from the NYCTF program generally teach in high-need urban schools throughout New York (Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2006). The mission of NYCTF is “to recruit and prepare high-quality, dedicated individuals to become teachers who raise student achievement in the New York City classrooms that need them most” (NYCTF, 2012). Thus, it is not surprising that NYCTF teachers would hold a positive social justice disposition. Furthermore, TEAM participants were religious Orthodox Jewish teachers who held traditional Orthodox values. As earlier stated, an example of their traditional beliefs is TEAM teachers required their classes to be separated by sex for religious purposes. It is possible that religious beliefs and culture contributed to the differences in beliefs about social justice in the classroom. These results should be interpreted with caution given the imbalance in sample size due to teacher availability.

It was found that teachers felt positively about incorporating diverse cultures and experiences into classroom lessons and discussions; self-examination of attitudes and beliefs about race, class, gender, disabilities, and sexual orientation; and teaching students to think critically about government positions and actions. While it is important that teacher educators continue to encourage teachers in these areas, it is more important teacher educators work with teachers in areas in which they felt less positively. Teachers felt least strongly that they should prepare students for life outside of the lives they will most likely lead; student success is contingent upon external variables outside of student hard work; and teachers did not feel strongly that their jobs were to be agents of societal change.

Teachers who focus on the lives they expect for their students reduce the possibilities for the students, which is a major issue in teaching for social justice because teachers may possess lower expectations for students of lower socioeconomic status and underrepresented groups. Students from these groups often are not exposed to adults from a wide range of career opportunities (Nakkula & Toshalis, 2006). Teacher educators need to work with teachers to help expand possibilities for these students. One method is to help the teacher focus on the individual’s aptitude; rather than assume that because a student is from a less affluent family, it is not worthwhile exploring future options.

While hard work is a critical variable for student success, it is certainly not the only variable. Other variables significantly impacting success are teacher quality, parental support, access to resources, classroom environment, classroom equity, and poverty, among others. It is important for teacher educators to help teachers not place blame on students for their lack of success when other variables could certainly be contributing factors. While it certainly is important for students to be held accountable, it is equally important to consider that students from less affluent urban areas do not enter the classroom with the same opportunities as students from wealthy suburban districts.
Jonathan Kozol said students in the wealthiest suburbs of New York receive three to four times the amount of funding toward their educations as do students in the South Bronx (Anonymous, 2000). Furthermore, the wealthier students are receiving two to three years of full-day preschool, which means children in less affluent schools are already behind their wealthier counterparts (Anonymous, 2000). Placing the blame for lack of academic success solely on the students is irresponsible and unproductive. Teacher educators must help teachers understand the complex issues surrounding student academic success, and find ways to increase the likelihood of student success.

Teachers are in a unique position to be agents of social change directly in the classroom, which means they can influence change outside of the classroom through the effects they have on their students’ lives. As stated earlier, Gutstein and Peterson (2005) said teaching in a neutral manner is not possible. All teaching is political in that what is omitted gives the message that certain issues are not important. Teacher educators must help teachers understand the impact they have on students, which can have profound effects not only on individual student lives, but also on community and society.

This article began with the notion that mathematics is a “gatekeeper” for quality college education and careers (D’Ambrosio, 2012; Gau Bartell, 2012; Gonzales, 2012; Koestler, 2012; Leonard, 2008; Martin, 2002; Moses & Cobb, 2001; Stinson, 2004), and subsequently quantitative literacy is necessary for some of the highest paying and most satisfying careers. Social justice considerations in education necessitate mathematics be of high priority in order for students from underrepresented groups to have the highest likelihood of later educational, career, and life success. The literature demonstrates that teachers are more likely to adopt a social justice perspective if there is consistent emphasis on social justice throughout teacher preparation programs (Koestler, 2012; Nieto, 2000). Teacher educators must support teachers in gaining a social justice perspective in their mathematics classrooms and to implement that perspective in a successful manner.
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Teaching Math to Young Children—What is Behind the Profession?

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Introduction
Some may view it as unusual for pre-service educators not to like math. It is not. In order to understand why this feeling exists, the article will take you on a journey of the pre-service teacher. Initially, the foundations of math acquisition will be explored. In this section, encouragement, learning and effective instruction will be discussed. Next, difficulties and factors of learning will be investigated. Then, the pre-service educator will be deconstructed. A closer look at how their self-efficacy and mathematical knowledge impact their role as future educators will be employed. Finally, many of the above elements will be applied to a community college’s methods course assignment in summative arrangement.

Foundations of Math Acquisition
The journey of the pre-service teacher begins with their own acquisition of the subject. In order for the expedition to come full circle, the initial stage should be reviewed. The foundations of acquisition incorporate encouragement, learning and effective instruction. 

**Encouragement**

Encouragement is the initial opportunity for mathematical acquisition. Parents play an important role in shaping the mindset of students. Carol Dweck believes that an optimistic attitude is what encourages some students to be persistent when others feel defeated. The child with a set state of mind feels there are a certain set of talents they were born with and stop trying when they embark on challenges. They do this because they believe they have reached their limit based on their natural abilities. Then there are students who have a growing mindset who do not apply limits to their potential. These students see challenges as an opportunity to improve. They have an inherent belief that experience and persistent can build their intelligence and rise to the challenge (Dweck, 2007). Modern research demonstrates inbred mathematical talent may not be as significant as we think, hard work is what is essential. Colvin (2008) supports this theory by stating focus and effort can improve skills. It is the hard work that impacts the success of a student in math. Parents are an essential aspect of the development of a child’s attitude towards math. Children come to believe in their potential to conquer challenges with focused effort. If a student is placed in a math class, encouraged and supported in working hard, both authors claim the accomplishments are limitless (Colvin, 2008; Dweck, 2007). Without individuals who are encouraging, children can surrender to the notion that if something is difficult it is not valuable (Colvin, 2008).

When young children are in primary school, they state math is their preferred subject. During these years, teachers adhere to assembling students in groups according to their abilities. Then when in middle school, they come to believe that success is inherent and those who can compute examples at an accelerated rate are successful. Finally, these negative dispositions are prolonged throughout high school and continue in college. These beliefs about academic performance are learned and maintained in environments where no one tells or demonstrates otherwise to this group of students. Teachers are essential in providing support and strategies that engage students in rich mathematical thinking. When students are engaged in these types of environments, they come to believe that success comes from working hard to comprehend math (Colvin, 2008). Encouragement and motivation are important elements of math acquisition. Five chief factors which influence motivation were found during a 20-year summary of motivation research. These elements are motivation is learned and pivots on students’ beliefs of their capacity to succeed or fail. Additionally, inherent motivation is better than a tangible incentive, inequities are impacted how groups receive instruction, and teachers are an important element (Huetinck & Munshin, 2008).
Learning

Experience is the most significant resource in self-efficacy. Students should not only be inspired by positive experiences, but those where they persist through challenges. Positive facilitators structure activities for triumph and persuade students to employ in self-evaluation. Evaluation is not only derived from the instructor, nor should it include only positive feedback. There should be an opportunity for growth and recognition of the process. Informal knowledge is knowledge children acquire in regular environments before attending school. Students have a sense of counting and numbers essential for formal math learning. This informal knowledge exists in older children and is developed through their experiences. These are the very same occurrences that can be utilized in developing official math knowledge. Teachers evaluate children’s prior knowledge to connect to new math learning (Gurganus, 2007).

Teaching concepts within a framework has the advantages of gaining interests and imaginations of students and providing mathematical knowledge in applicable relatable situations. The ultimate strategy is an awareness of students’ interests and how to challenge and support them. When students are able to apply math to everyday life, they are more receptive because they find value in learning concepts. They often ask, “When are we going to use this?” Therefore, including circumstances from students' personal experiences incorporates interest to the topic being studied. Passionate teachers who display an adoration and knowledge of math, anticipate student success, and utilize wide-ranging problem-solving techniques are those who motivate students to ascertain academic merit (Huetinck & Munshin, 2008).

A report on primary education references the need for students to develop responsibility for what and how they learn. They should be aware that knowledge is not only transferred, but negotiated and re-created. The activities presented to young students should provide chance to take control of their learning and be empowered in that ownership (Sangster, 2012). A student’s belief in their capabilities can impact how they think, their motivation, actions, level of attempts, insistence, and level of achievement. Elevated self-efficacy and improved performance result when they set undersized goals, relate precise learning strategies, and receive performance-dependent rewards. Being able to assure themselves as a result of their beliefs are essential components to handling challenging tasks. However, when students encounter failure and see it as being out of their power to manage, learned helplessness can decline their performance (Swar, 2005).

Huetinck & Munshin (2008) describe an elementary research study where children were not told if their answers were correct. This purpose of this was to foster independent thinkers. Some of the children declined additional supposedly because he was not conducting himself as a teacher and disclosing or verifying responses. Over time these children became accustomed and began to verify their own work. They did not rely
on the instructor. As a result of this research some ideas emerged—math requires proof and computation. They realized math problems do not always contain information necessary for solutions, sometimes they had to use prior knowledge to obtain solutions. In addition they found math to be useful and integrated with other topics (Huetinck & Munshin, 2008).

Effective Instruction

Integration is also seen in previously-learned skills. Moreover, progress in math extends greatly on this incorporation. Therefore, it is essential for instruction to be clear and methodical with essential prerequisite skills taught at the forefront. For example, expectations of memorization of addition facts should not exist when students have not understood the basic notion of addition or associated symbols. Adequate practice while students acquire skills is helpful. This practice should match the needs of each child. Some students require manipulative, whereas others may not. It is totally dependent on each student and their needs. As students progress into the upper grades, they are generally tracked based on their abilities. Intensive remediation of basic math skills is essential and usually provided to those who require the services. It helps those students ascertain these skills which are needed to function in everyday life, many occupations and is a gateway for higher education (Gurganus, 2007).

Active learning is cornerstone in assisting a larger number of individuals learn math successfully and providing venues for them to study the subject at a deeper level. This is accurate within the higher education sector. This type of learning can assist educators in higher education respond to the agendas of learning, teaching and increasing effectiveness. It can also assist with some problems ensued in the transition to higher education. Additional outcomes of incorporating active learning is motivation, enthusiasm, excitement and a maintained admiration of math for students and educators (Kahn, Kyle & Institute for Learning and Teaching in Higher Education, 2002).

Very little research about effective components of math instruction is known when compared to that of reading instruction. Nevertheless, information can be deduced from the excellent research that is in existence in conjunction with the standard grade-level expectations in this subject. There is no single approach that works best for math as can be said about reading. Instead, the focus is on essential skills associated with learning math which should be concentrated on during math instruction. The significance of these skills vary across elementary and secondary grades (Gurganus, 2007).

Challenges in Math Acquisition

The journey of the pre-service teacher continues with obstacles that could be encountered during acquisition of the subject. For some acquisition comes naturally, whereas others have more difficulty. These are the challenges that prevent individuals from building on
motivation and positive experiences. Although all pre-service instructors may have not personally encountered these challenges, they may in fact have students who will experience them. Some challenges of acquisition include environmental and individualized factors.

The subject of Math has been classified as the porter of triumph or failure for high school graduation and occupation victory. It is important that this subject become a force as opposed to a strainer in American education. A deficiency in mathematical skill and understanding affect an individual’s aptitude to make significant life, career and academic decisions (Richardson, Sherman & Yard, 2008). Students fall below their projected level of math attainment for many reasons. Some students have expressed that they had no comprehension of the subject matter or it was not a relatable subject. This expression is the basis for two factorial categories—environmental and individualized factors (Richardson, Sherman & Yard, 2008).

Environmental Factors

Instruction should be taught through integration where there are concept-building opportunities, real-world applications and a spiraling curriculum. When math is relatable and concepts are repetitive, they have an opportunity to understand. Isolated facts are foundation for disconnection. In addition, the learner’s ability should be considered. Modifications should exist for those who are not able to grasp content initially. Then when the curriculum repeats itself, they can try to understand it again with the encouragement of the teacher and in earlier years, their parents (Gurganus, 2007).

Individualized Factors

Some students feel that because they are lucky, they succeed in math. These external beliefs are not based on their general comprehension or effort. In addition, they may not feel that they were born with mathematical strengths. These views stifle their ability to persist. Some students lack well-developed mental strategies relative for procedures, organization or recall of facts. Also focus, ability to deal with long-term projects and isolated facts also create interference. Meaningless memorizing terms without association is not helpful. Insufficient concept development can be evidenced in older students. Sometimes educators assume mastery and the foundational concepts and experiences are missing. Thus presenting challenges in learning new concepts that originate from the eliminated concepts (Gurganus, 2007).

Pre-Service Teachers—The Journey Continues

The journey of the pre-service teacher continues with their actual experiences in the classroom. Through challenges and foundations in acquiring mathematical concepts, future educators are provided opportunities that enable them to function as successful educators. They gain experience that can be utilized in their interaction with young
students. Many thoughts fill their head as to the adequacy of their own mathematical knowledge, self-efficacy and disposition toward the subject based on their experience with the subject thus far.

**Teacher Efficacy**

Mathematical instructional strategies, past math experiences, teacher implementation of significant academic strategies and pre-service educators’ influences on notions of teaching effectiveness were associated with math teacher efficacy. Highly efficacious educators were stronger math teachers than those with a lower sense of efficacy. There are two factors in a teacher’s self-efficacy. One of which is personal teacher efficacy—their belief in their effectiveness as an educator. The second is a positive outcome they anticipate in their students’ learning despite external elements. Teacher efficacy also has been linked to notable variables like classroom instructional strategies and readiness to embrace modernizations. Educators with a high sense of efficacy are more likely to utilize student-centered, try innovative strategies, and inquiry approaches, whereas educators from the other side of the spectrum would rely more on traditional methods like lecture and direct instruction. In the limited amount of studies on math teacher efficacy of elementary pre-service teachers, significant increases in math efficacy were correlated to pre-service teachers’ participation in math methods courses (Swar, 2005).

**Teacher’s Mathematical Knowledge**

Pre-service elementary teachers stated feelings of anxiety and felt the desire to utilize similar strategies to those they experienced during their education. Yet, education of these individuals is meant to decrease the level of trepidation as other instructional strategies instructional approaches are introduced during their academic career in higher education (Levine, 1998). During methods courses, there is a concentrated focus on specific mathematical concepts. Research indicates that there was a significant improvement in pre-service teachers’ understandings. However, there continues to be deficiency of accord in the literature as to information educators should know about math to disseminate it well (Kajander, 2010). During teacher education programs, there are opportunities which support elementary pre-service teacher development.

Scholars dispute successful teachers must comprehend more than the act of instruction. Some propose that student teaching placements can be considered learning laboratories where student interns become familiar with higher education and the school. However, some say that this is not a positive environment for pre-service teachers to picture a successful educator. (Mewborn, 1999).

Nevertheless, research demonstrates that college students might become more engaged in their pre-service coursework if they perceived that these assignments were
instrumental for their upcoming classroom instructors. Therefore, they must first have a detailed awareness of what their future as educators will be like (Moyer & Husman, 2006). When students have meaningful experiences, that which they can recall later, they become more engaged in methods courses (Moyer & Husman, 2006).

Pre-Service Educators at Hostos

Lesson plans flood the traditional environment of our community college method courses. The philosophy behind such practice stems from that of a constructivist view. Participants should build their knowledge in order for it to be most meaningful. Sometimes students try their lessons on children that they know or formally during their field work class. For the purpose of this assignment, I will focus solely on a Math and Science course. This is a methods course designed to prepare future teachers to teach math and science to young children. In the course, they become aware of theory and are able to connect it to practice in front of their peers. Hybrid classes are opportunities to balance the in-person lecture of a traditional setting and independent online atmosphere. Originally, I thought in a hybrid setting, it would be a bit challenging to provide an equivalent experience. However, I found that the online setting lent itself to a much richer experience in lesson planning.

The following assignment was designed for teacher-education students to put theory into practice. For several weeks, elements of the lesson plan were being taught through the traditional lecture. Resources are provided for lesson-plan development. Within a handbook, students can review sample lesson plans, explanation of lesson-plan elements and a rubric. This information is not presented all at once. Perhaps two elements are taught at a time in conjunction with practice. So the lesson plans that they create only incorporate the elements that were reviewed. As new elements are introduced during the lecture, they are included in the students’ lesson plans. A wealth of resources and opportunities are provided. They are considered tools. Yet, ultimately it is up to the student to utilize those tools. The assignment is entitled “Lesson Plan Transcription and Addition”. The guidelines are as follows:

Locate and examine a YouTube instructional video in the area of Math. As you watch the video write down each part of the lesson plan that you viewed (i.e. motivation, grade level, evaluation…etc.). Then write which elements (if any) were missing from the video. Finally, write a follow up for the lesson. What would you teach this group of students next? This is based on their response to the instructional session. Do you feel anything needs to be re-taught? Do you feel they are ready for the next aspect of that topic? What would that be?

Students are asked to produce their findings in a typed report which includes the transcribed lesson, justification of the follow-up activity and citation of the YouTube video. An option (not a requirement) for the more technologically-adventurous students is
to present that information in a mind-map format. There are several free software packages available. On the day the assignment is due, students give a brief presentation of their lesson plan to the entire group. Yet, in cooperative learning groups, neighboring students can offer alternative strategies for follow-up activities. This is a more intimate way of sharing, respecting the feedback of others and feeling empowered to plan and interpret lesson plans. In addition, they are able to observe actual children learning, witness how they respond and strategize ways to support their learning. As a result of this assignment, students are able to utilize both the online and traditional environment. This is a task that will be required in their future and they can begin now.

Context, Preparation, and Objectives

A fundamental learning objective is for students to experience meaningful learning that they will present to their students. Students scrutinize the lesson of an educator in the field. In doing so, they demonstrate higher-level thinking. Students are able to analyze, synthesize and evaluate a lesson. When eliminating themselves from the picture, they maintain a more objective view as opposed to that of a biased nature. They tend to see more errors from an external examination. This is one lesson of many required for a larger assignment—the lesson plan portfolio. This is a collection of lesson plans written during the semester. Initially, the lessons are simplistic as they incorporate few lesson plan elements. Then they include all elements. They begin to exchange their created lessons with classmates and provide feedback. Next this assignment allows them to view the lesson of an educator in the field, analyze and extend their lesson. This assessment is a direct result of their lesson development and exchanges. The feedback has a different tone when it comes from the professor as opposed to a peer. At the beginning of the process, some students question the authority of a peer revising their work. However, I reassure them that revising the lesson plans of others is essential to the process. Practice makes perfect. In addition, the resources assist in the process—especially the checklist. Therefore, their colleague is centrally ensuring certain elements are present and they understand the lesson and the practicality. Choosing a video that lends itself to many elements of the lesson is essential. The option to transcribe the lesson via mind map software or typed is another helpful element. The hybrid course is a transition for many into an asynchronous course. Therefore, baby steps and gradual immersion into technology is helpful to those who may not be ready. In addition, I review how to create the mind map during face-to-face lectures weeks prior to the due date. Some students are willing to attempt the mind map. Others are not there and I realize that. The primary objective of the course is that they are able to create lessons that are appropriate for young students in the area of math and science.

Summary

The initial mathematical acquisition which occurs in the earlier years of instruction is paramount in later years. This includes encouragement, learning and effective instruction.
Students may encounter challenges during acquisition; those can be mastered whether they be environmental or individualized. The experience as a pre-service educator provides an opportunity for success with several elements mentioned during their overall journey. Efficacy and their mathematical knowledge are considered. Below review these elements as we consider the college method course assignment:

<table>
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<th>Foundation</th>
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<th>Pre-Service</th>
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<td>• Factors (environmental &amp; individualized)</td>
<td>• Teacher Efficacy</td>
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<td>• Learning</td>
<td>• Concept building</td>
<td>• Teacher’s Mathematical Knowledge</td>
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<td>• Effective Instruction</td>
<td>• Cyclical lessons</td>
<td>Mathematical Knowledge</td>
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<td></td>
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<td>• Meaningful learning</td>
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<td>• Opportunity for application</td>
<td>• Reinforced efficacy through presentation and feedback</td>
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<td>• Higher-order thinking</td>
<td>• Immersion in lesson plans</td>
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<td>throughout course and additional methods courses</td>
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**Bibliography**


LEARNING ROUTES METHOD:
how to build stronger connections
between learners and their learning goals

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Abstract
Research efforts show that several styles and approaches to learning exist: each person learns in his/her own way and learning performances may significantly increase or decrease according to the attention given to this issue. Each person has particular cognitive and emotional characteristics. And it is easy to agree upon the fact that these are strongly tied together, both in space and time. In this scenario, an issue arises very frequently: how can I, as a parent or teacher, educator etc, address this problem while considering all the constraints I encounter? Limits might be determined by: the economic budget, the time and space allowed for working with students, more or less ability of the educator to establish a fruitful communication with the student, and vice-versa. This article addresses a practical and easy-to-use way to take appropriately into account the initial intellectual condition of a person, whenever a teacher, educator or parent has to facilitate learning of the person in a certain direction.
Introduction

After the pre-school age, a person must usually enter a long period of directed, institutional curriculum of learning, which can also span 12 years or even more. A child gets into the school stage with a large range of ideas, attitudes, radical ways of approaching the world, activities, peers, a family and general background environment. This background heavily affects the way he/she thinks and learns. In many situations, this intellectual heritage is almost un-modifiable (Gardner, 1993). Sadly, the meeting between this personal background and the rigor of the curriculum creates situations where the student can be classified as lazy or affected by an attention disorder. In extreme situation, he might even be unable to attend school anymore (Levine, 2003). Many solutions are out there to get rid of this problem (Levine, 2006), even if most of them remain today unexploited.

Each country (or even a group of countries together) decides what must be taught (and hopefully learned) or may decide to provide a less strict range of learning goals. In this case, school educational professionals can decide upon topics interesting to learners (National Council of Teachers of Mathematics [NCTM], 2000; European Parliament [EU], 2006). In a sense, the situation is almost everywhere the same: a teacher in front of a number of young students. The former says something must be learned, the latter agree partially, they do not frequently agree at all and they are very rarely enthusiastic about the goal proposed.

This article does not address the topic of what should be taught inside educational institutions, but it proposes a way of how a young mind may be invited to get in touch with curriculum topics chosen by the school, the teacher or whoever is in charge. The proposed method addresses mainly situations where learning may be slow and difficult. One of these cases is given by mathematics curriculum. Hence the examples shown at the end of the paper deal clearly with mathematics.

As a side note, there is a huge amount of scientific evidence that learning evolves better when the student is actively interested and involved in the process. The reader is referred to the famous works in Bruner (1960, 1966) and Montessori (1986) and bibliography therein. Anyway, arranging curricula to meet students’ needs is not always possible. That’s the point where this paper comes onto the stage.

The rest of this work is organized as follows: Section 2 explains the method proposed, its main steps and implementation. Section 3 shows a few possible examples of application to students. Section 4 draws conclusions and proposes possible future research efforts.

2 The Method

A general learning situation can be summarized in this way: a student is close to face a new topic (where the topic may be theoretical in nature, practical, or an ability, a mind habit or else). Of course, this topic is not already inside the mind of the student, or it can be there in a somewhat larger or smaller extent and depths. Anyway, an improvement is advisable, for a number or reasons, depending on the specific situation.
Let’s place some starting questions to tackle this issue:

(a) how far is the student mind from the topic? And how can we better describe this “far”?

(b) as above stated, if we suppose the topic must be taught, how can an educator build an effective connection between the topic and the student? And, very importantly, how can we build a practically usable connection, not only a theoretical one? By “practically usable” we mean a learning path which is viable for the student as well as the educator. Education at its core should be regarded as meaningful especially by the student.

2.1 Step 1: The Internal Home of a Learner

A student always starts every learning attempt moving from his/her own internal home: as addressed in a variety of research studies (Gardner, 1993; Levine, 2002) and bibliography therein, what we call “internal home” is the entire set of mind habits, memories, social and cultural heritage or traditions, personal theories about the world, other people in the world and family, things and fundamental philosophical issues a person always carries inside, whatever he/she does. Moreover, not only the starting condition of a student differs that way but also the learning approach he/she has to the new knowledge to be gained might be very different. This is famously summarized in the groundbreaking work of Gardner (1981) and more recently in his work of 2006. So, how can we represent this internal universe of a young student in a practical way? Usually, teachers have little time to pay attention to this fundamental problem, even when they are really aware of its importance.

Our first proposal here is that the teacher (or the educational staff as a whole) should try to represent the internal intellectual environment of the student using concept maps [Novak, 2010, and bibliography therein]. Concept maps are a graph-like tool which has very good characteristics to allow both a deep understanding of the subject studied (be it a person, an idea or whatever else) and a gentle learning curve to begin with.¹

Briefly, a concept map has two key features (Novak, 2006): a central concept to be analyzed and a focus question which should drive the attention both of the concept map builder and the concept map reader.

So, how to construct the concept map of the student internal home?

A widely used tool in educational settings is useful for this goal. CmapTools² collects many of the features needed to depict the internal home of a learner in an easy way to be implemented. Here we propose two ways this can be accomplished, but of course many others may arise in the future:

(a) the teacher explicitly asks each student to realize a concept map which represents him/herself: this is a key step and can also arrive after a short initial training of

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¹ At the moment, other graph-like tools seem appropriate for the goal but they have not been used yet by the author.
² Available at http://cmap.ihmc.us
students to get acquainted with the tool (very easy to use, honestly). The student should be invited to place him/herself as the central concept of the map. The map can show the following as focus question:

• “what would you like to spend time for – say weekly – if you could decide your weekly planning entirely on your own?” (this way of placing questions reflects the assumption well known in pedagogy – and in everyday life – that humans learn better topics they are more interested in)

• “try to describe yourself showing attitudes, qualities, family, friends, hobbies and so on; try to describe your projects about your desired future job, which kind of people you would like to be surrounded by or work with or play with or live with etc.” Of course, these questions may heavily vary with respect to the age and other personal conditions of the students. A similar approach to the one here is depicted in Barringer (2010) where the authors suggest the following questions: 1) if you were to design your desired day, what would you be doing? 2) what parts of school are easiest for you and why? 3) what are your affinities – those things you love to do or learn about?

(b) the teacher devotes a certain amount of time to build concept maps of the students' internal homes. We think this way is slower and moreover, has a fundamental lack: it describes the students’ internal world the way the teacher sees it and not the way the students do. This is very prone to misunderstandings and might lead to failures. We understand, however, that time constraints in today educational settings are strict and so this second way can be more viable than the former.

After a while, the internal home of the student is described by a concept map. Taking advantage of the graph-like nature of concept maps, we suggest to represent them via the usual graph notation: $MH = \{V, E\}$ where $MH$ stands for the Mind’s Home of the student, $V$ is the set of all the concepts the students placed in the map, and $E$ is the set of all the links the student drew to connect concepts one another. As far as this notation regards, CmapTools offers an appropriate range of built-in functions which allow to group together and list concepts and linking phrases.

Before going any further, the teacher has to sketch a concept map of the topic he/she wants to teach, as large and detailed as possible. This will be addressed as Topic Concept Map (TCM hereafter).

2.2 Step 2: The Route to the Topic, or the Learning Route

After the first step, the staff has the student’s mind concept map. In a sense, the teacher has a deeper “knowledge” of the student and this knowledge is easily and conceptually schematized.

Now what? Let’s address a frequent and also to the least desirable situation: the topic to be taught is not inside the MH graph of the learner. Otherwise stated, in a less
formal fashion, the learner does not ”see” it in his/her practical and theoretical world of interests. Now, irrespective to the hardness of the situation, the concept to be taught is, in the end, a ”concept” on its own. So let’s place it in the MH graph as an isolated node, i.e. it has no link with anyone of the concepts sketched by the student. The second and key step of the method must now be undertaken.

We recall a somehow strict similarity to the backward problem solving process described in Polya (1981): you have a mathematical problem carrying some data and you should get to a solution, which is not directly linked to those data (where directly means with just one logical step). Otherwise stated, you have to build a multi-step connection between the desired result and the actual data you have. Our analogy with the educational problem comes clearer now: the student MH cannot actually be one-step connected with the topic but it might be with a multi-step process. How can these subsequent steps be built? In order to solve this issue we define here the second key concept of our method.

Taking another similarity with a well-known concept in psychology, the “Zone of Proximal Development” by L. Vygotskij (1986) we use here what we call the Node of Proximal Learning (NPL, hereafter). Taking whichever of the nodes (= concepts) written in the MH, the student may be now asked to list topics which he/she is interested in learning, connected to those concepts inserted by him/herself.

Of course, new concepts may appropriately be proposed by the teacher—to students, to check their reactions. This will eventually lead to a set H made by lists of ”desired” topics. The teacher has now to choose the most appropriate topics from those lists, add them to the MH map and see if any of those topics belong to the topic to be taught (or, stated in set like language, if the intersection between the TCM and the MH maps is now not zero).

Two situations may arise:

(a) the intersection is not zero, hence the teacher and the student have built together a meaningful multistep connection between MH and TCM. The educational process begins with the very first step where the student and the teacher now agree upon

(b) the intersection is still zero, hence two steps have to be undertaken: the teacher tries to reformulate the TCM taking seriously into account the \( \{MH + H\} \) set he/she now has. On the other hand, the student tries to do the same with his/her MH + H concept map. After that, the teacher searches again with possible intersections between \( \{MH + H\}_2 \) and TCM

(c) the process goes back to (a) again, until a non-zero intersection arises. After a non-zero intersection is found between \( \{MH+H\}_i \) and TCM\(_i\) at the \( i \)-th step of the process, a multistep connection between the original MH and TCM is done. We would like to stress here that this connection is now meaningful, simply by construction, since it has been built both by the teacher and the student, who has enlarged its MH step-by-step until getting in touch with the borders of the TCM, reshaped by the teacher. The topic to be taught should make now sense to him/her, since it belongs to its enlarged \( \{MH+H\} \).
2.3 Observations

As side notes to the backward learning method, we focus the attention on a number of points:

- the graph approach, implemented via concept maps, give almost immediately a quantitative analysis of the conceptual distance between the usual MH of the student and the TCM studied. Otherwise stated, a teacher may also measure that a TCM is simply one-step away from the \(\{MH+H\}\) and so the process of building a meaningful connection between the two maps is really easy. On the other hand, if the intersection between \(\{MH+H\}\) and TCM is still zero after a long run of iterations in the above process, the connection will be very likely weak and effective learning more and more difficult

- the curriculum may be reshaped, year after year or semester after semester, in order to reduce the multistep distance between \(\{MH+H\}\) and TCM, hence assuring a finer grained and especially more meaningful learning for students, since they will find nearer TCM’s to their actual MH’s

- MH’s have to be periodically reshaped, especially for younger students, since their experience of the world may frequently undergo big changes and this implies careful changes in the teaching approach; of course, the MH’s need not to be done again each time a new topic is to be taught

- the method proposed also gives educational staff the opportunity of building a visual portfolio of the students, where the changes of their cognitive attitude toward curriculum topics can be easily tracked

- taking advantage of statistical tools, it is also possible to count how many MH’s are 1-step away from the TCM or 2-steps away and so on. The larger the average distance between MH and TCM and the harder the job for the teacher

- it is here noteworthy that the proposed learning method addresses at least three out of four of the main characteristics listed by J. Bruner (1966) about a theory of instruction (ToI hereafter). Of course, the method proposed here is definitely not a theory but it is strictly linked to it. They can be summarized here:
  1. a ToI should say which experiences are more suitable to generate in a learner an attitude for learning, being it general or particular in nature
  2. a ToI should specify the way a part of knowledge is to be structured such that it can be readily learned
  3. a ToI should address the optimal progression to show the topic to be learned

Emphasized words highlight the connection with the method proposed here. Experiences refer to MH, structure refers to TCM and progression refer to the learning path. All these issues are addressed in this work.

3 Implementation

We propose here a summarizing chart where the method can be overviewed in its main steps:

1. assessment of the starting internal condition of the student/learner:
this is accomplished via the generation of a concept map of the MH (Mind Home) of the student where his/her main personal characteristics are schematically depicted; the concept map realization is better up to the student, since it will mirror his/her way of thinking

2. making the TCM (Topic Concept Map) of the topic to be taught by the teacher/educator:
   this TCM is better assigned to the teacher/educator, since it mirrors the way he/she sees the topic and so it will eventually highlight important differences between students’ and teachers’ way of thinking

3. searching of the conceptual route between MH+H and TCM via subsequent nodes linking (called Nodes of Proximal Learning) eventually building a meaningful learning route.

3.1 Examples

We show here a possible situation where this method may be useful. Of course, more examples will come from experimentations on site, like in schools and many other educational places.

Let’s assume the topic to be taught is “cartesian 2 dim space and its coordinates”, which in turn belongs to the very general topic of cartesian geometry. First of all a student has to build his/her own conceptual map, where he/she places things considered interesting in his/her way of seeing life around.

Here we report two self-concept maps built by students of secondary schools, aged 14. Of course, maps can contain conceptual or logical errors, but the author decided not to correct them because these errors should be taken into account when trying to understand the way a student thinks. White nodes represent the MH of the student, orange nodes represent the TCM and green nodes highlight the learning route between the two maps above.

Of course, two students are not a statistically relevant test set, but the NPL’s routes found may be very useful in undertaking the activity with those students. A whole range of educational experiments should be started in order to gain a deeper understanding of this method. Concept maps were made by students on their own.

The MH of the two students is composed of all relevant nodes they put into the map. The distinction of a relevant node from a not relevant one is subtle, and it may depends on a variety of factors: the TCM, the teacher point of view itself, possible prior interviews to the students which may integrate well with the actual map built and add new knowledge about the students, etc. We will strictly follow now the protocol proposed in Section 2, hence the key steps are as follows:

4 Maps have been translated into English, since they were written originally in Italian. No nodes have been hidden or modified, neither during nor after the completion of the map by the student. Only personal names of the students, of their friends, of their relatives and other personal data like the school or the city have been modified and are totally fictitious.
1. initial assessment of the starting internal condition of the student/learner, i.e. the learner’s MH

2. realization of the TCM (Topic Concept Map) of the topic to be taught by the teacher/educator

3. searching of the conceptual route between \{MH+H\} and TCM via subsequent nodes linking (NPL nodes), eventually leading to a meaningful learning route.

**Student 1**

**step 1**: initial assessment of the starting internal condition of the learner.

Please, note that different concepts are separated by a slash “/”. In this context, we will highlight the possibility of linking the MH with cartesian geometry.

MH = \{ student 1 / football / building and programming robots / youngest son of his father / brother of John / my school first grade / swimming pool / sports / talking with friends / go out on saturday nights \}

**step 2**: realization of the TCM (Topic Concept Map) of the topic to be taught by the teacher/educator. See the nodes highlighted in orange in figure 1

**step 3**: searching of the conceptual route between \{MH+H\} and TCM via subsequent NPL nodes.

The most suitable node to be taken into account is the one called building and programming robots. The teacher may ask the student if he/she may be interested in programming a simple code to drive the robot around a certain zone, following a desired path. In case of acceptance, a new NPL is found and this will lead to a new node called “simulation routes” or “simulated routes on a field”, that can be used to determine the following learning route to cartesian geometry:

\{“simulated routes on a field” - also called - “trajectories” - described in terms of - “cartesian coordinates” - is used by - “cartesian geometry”\}

So, in this setting, we may get a multistep connection between the student and the topic, via realistic intermediate learning goals, highlighted by green in the figure. The green and orange concepts are added after the initial concept map has been built, i.e. they refer to sets like \{MH+H\}.

Here we report for the sake of completeness the new and final set \{MH+H\} where concepts belonging to the set H are reported in italics.

\{MH + H\} = \{ student 1 / soccer / cartesian coordinates / cartesian geometry / cartesian plane / circles, ellipsis, parabolas, hyperboles / building and programming robots / youngest son of his father / brother of John / my school / first grade / lines / many geometric objects / swimming / simulated routes on a field / sports / talking with friends / strange curves / trajectories / go out on saturday nights \}

**Student 2**

Let’s now turn our attention to the second student’s map.

**step 1**: initial assessment of the starting internal condition of the student/learner
MH = \{ \text{student 2/ big nose / friendly / listen to music / basket / clear / brown / chatting / computer / body /} \\
\text{enjoy my life / sleeping / facebook / playing / Tom / watching interesting videos / play role / inventions / slim / eating / fashion / not enough time to make things / news / taking rest / playstation / sports / record / normal / always moving / talk about myself / happy / joker / my computer / talking with friends / being stuck / studying / willing / dresses / face } \}

\textbf{step 2}: realization of the TCM (Topic Concept Map) of the topic to be taught by the teacher/educator. See the nodes highlighted in orange in figure 2.

\textbf{step 3}: searching of the conceptual learning route between \{MH + H\} and TCM via subsequent NPL nodes. In this case, finding a reliable route made of subsequent NPL’s has been somehow harder. But the author is confident it may be useful also in this situation. Here we report the route built:

\{\text{field - technological - like - ”robotics” - useful also for - building extraterrestrial probes - goal - routes - by - moving autonomously in a new planet - defined by - programmed coordinates with computer languages - uses - cartesian geometry}\}

For the sake of completeness, we report the new and final set \{MH+H\} where concepts belonging to the set H are written in italics.

\{MH+H\} = \{\text{big nose / friendly / listen to music / basket / clear / cartesian geometry / brown / chatting / computer / programmed coordinates with computer languages / body / enjoy my life / sleeping / facebook / playing / Tom / watching interesting videos / play role / inventions / slim / eating / fashion / moving autonomously in a new planet / not enough time to make things / news / taking rest / routes / playstation / sports / building extraterrestrial probes / record / normal / robotics / always moving / talk about myself / happy / joker / my computer / talking to friends / being stuck / studying / technological / willing / dresses / face } \}

Again, white nodes belong to the MH of the student, green nodes belong to the learning route built and orange nodes belong to the TCM. A final remark is important for learning routes: they have been built using robotics concepts in both cases. Of course, a huge number of other connections can be made. They depend on the MH’s of students and can be constructed by many other tools like math manipulatives, computers, literature, music, science and so on.

4 Conclusions and Future Work

This work shows a practical way to build a meaningful learning route between the inner emotional and intellectual world of a student and a topic to be taught by a teacher. This method tries to be an easy to implement way to link students’ interests to fixed curriculum topics. This is especially desirable in many situations where it is neither possible to strictly follow the former nor to rigidly adhere to the latter.

This is a starting idea and a lot of work needs to be done:

- tests with a far larger number of students, in order to gain a better understanding of their learning differences and of their starting MH’s
• tests in a variety of settings, like primary and secondary schools. It is advisable to test the method also outside schools, e.g. universities and beyond, especially the corporate sector.

**Figure 1: Concept Map of the First Student.**
Figure 2: Concept map of the second student.
Reference List
The Simple Theory of Informal Rules

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Keywords: Statistics education; Informal statistical inference; Probability; Sampling variation

Abstract

The latest changes in the New Zealand high school mathematics curriculum include basic study of informal and formal statistical inference. Statistics education researchers have made a great effort to deliver these changes to the New Zealand high schools. However, informal statistical inference concepts currently adopted in New Zealand high schools are visually oriented and rule based, but not closely linked to the context. The aim of this paper is to investigate the relationships between the key probability and informal statistical inference concepts. Based on the simplified classical formal statistical inference we derive and critically evaluate the rules for the informal statistical inference currently adopted in New Zealand high schools.
1. Introduction

Modern technology enables students to easily learn the calculations related to the statistical inference as some user friendly interactive software like TinkerPlots (Konold and Miller 2005), are available now. However, research indicates that students typically have difficulties in understanding the statistical reasoning and statistical concepts. See for example, Batanero (2000); Saldahna and Thompson (2002); Chance et al. (2004); Castro-Sotos, et al. (2007); Liu & Thompson (2009); Harradine et al. (2011); Pratt et al. (2008).

Research indicates also that some high school mathematics teachers have the similar misconceptions and difficulties in the understanding of the statistical inference as their students do. See for example, Vallecillos (1999); Lui & Thompson (2009); Harradine et al. (2011).

A pivotal point in statistics education was in 1999 when SRTL (Statistical Reasoning, Thinking and Literacy) series of biannual international research forums were organized. To overcome the difficulties in the understanding of statistical inference, statistics education researchers at the SRTL – 4 forum came to a consensus that students should be provided with the opportunity of learning the informal statistical inference (Ben-Zvi et al., 2007).

As the importance of the statistical inference is increasingly recognized worldwide, its concept has become part of the high school curriculum in many countries. The latest changes of the New Zealand high school mathematics curriculum include the basic study of the statistical inference. For example, year 11 and 12 students (15 – 16 years old) learn sampling methods, sample distributions, sampling variability and informal inference; year 13 students (17 years old) learn confidence intervals and formal statistical inference (NZ Ministry of education, 2010).

The limited research, available about New Zealand teachers’ understanding of statistical inference, indicates that some high school mathematics teachers could share the same difficulties in understanding the ideas of the statistical inference as their colleagues around the globe. See for example Pfannkuch (2006) and Wild (2006).

Wild et al. (2010, 2011) have made a great effort to deliver ideas of informal statistical inference to New Zealand high schools. The concepts developed by Wild and his colleagues represent an innovative approach that has given invaluable support to teachers and their students in understanding the key concepts of the sampling variability while rightfully keeping the focus on the statistical enquiry cycle. The visual approach developed by Wild et al. (2010, 2011) (www.censusatschool.org.nz/2009/informal-inference/WPRH/) enables students to develop their understanding of sampling variability and make inference about populations from the single samples. As Wild et al. (2011) mentioned, the support around the concepts of the informal statistical inference
has been primarily targeted to the mainstream of students only – ‘We want to arrive at
creations of statistical inference that are accessible to the bulk of students and not
merely an intellectual élite’ (p.251, Wild et al., 2011). As a result of the target audience
mentioned above, the informal inference rules developed by Wild et al. (2010, 2011) are
empirically derived by simulating normal data, but not closely linked to the context.
However, many high school students have expressed a desire for a deeper understanding
of the connections between the informal inference rules and their existing understanding
of the key probability and statistical concepts (P. Doyle, personal communication, August
1, 2012; New Zealand Association of Mathematics Teachers, Hawkes Bay Branch
meeting, personal communication, November 11, 2012). We still have some students,
who prefer to know the basis for a rule rather than applying it without understanding.

Based on the simplified classical formal statistical inference in this article we unpack and
explain the theory behind the informal statistical inference rules currently implemented in
the NZ mathematics curriculum.

2. Basis for the rules of informal inference

The main concept of classical statistical inference is based on the fact, that if the values
of a variable of a population have normal distribution, the mean values of all possible
random samples of the population can be modeled very well by the normal distribution.
The mean value of a single random sample (µ_sample) varies from sample to sample and the
standard deviation of the sample means is inversely proportional to the square root of the
sample size (Kendall and Stuart, 1967).

It is impossible to figure out the exact value of the population mean if we have a single
random sample. Instead, we can give the interval estimate of values for the population
mean (µ).

In this paper we use the 95% confidence interval for the population mean – i.e. we use
the property of the normal distribution that 95% of the data lie about within the 2
standard deviations of the mean (McGill et al., 1978):

\[ \mu_{\text{sample}} - 2 \frac{\sigma}{\sqrt{n}} < \mu < \mu_{\text{sample}} + 2 \frac{\sigma}{\sqrt{n}} \]  

(Inequality 1)

When analyzing the statistical information visually, box plots have been commonly used
since they were introduced by Tukey (1970). To find the relationship between the
standard deviation of a sample (S) and its interquartile range (IQR) for a normally
distributed data, we use inverse standard normal distribution (Fig.1), which leads to:
IQR = 1.35S \quad \text{(Equation 2)}

**Figure 1.** For a normally distributed data there is a simple relationship between the standard deviation (S) and the interquartile range of a sample (IQR): \( IQR = 1.35 \times S \), where \( IQR = UQ - LQ \).

The inequality 1 and the equation 2 give the 95% confidence interval for a population mean

\[
\mu_{\text{sample}} - 1.5 \frac{IQR}{\sqrt{n}} < \mu < \mu_{\text{sample}} + 1.5 \frac{IQR}{\sqrt{n}} \quad \text{(Inequality 3)}
\]

The Inequality (3) is the basis for deriving informal rules of the statistical inference currently adopted in New Zealand high schools.

**3. Basis of making a call about two populations**

At a high school level we teach rules to make a call about two populations. By applying these rules students can make a call whether the population B tends to be bigger than the population A. However, we do not teach them - what does it mean that the population B tends to be bigger than the population A (or vice versa). For some reason, we have missed giving them the most important piece of information for making a call – the definition. We think that in teaching informal statistical inference, the starting point should be the introduction of the probabilistic definition for making a call, the definition for comparison of two groups. For example, the classical question: ‘do Year 11 boys tend
to be taller than Year 11 girls in NZ?’ involves in general three different types of comparisons:
1) Comparison of a particular Y11 boy’s height with a particular Y11 girl’s height. E.g. comparison of Adam’s and Emily’s heights;
2) Comparison of random samples of Y11 boys and Y11 girls. E.g. comparison of two random samples of 30 boys and 40 girls selected from the data base of www.censusatschool.co.nz;
3) Comparison of two populations. E.g. comparison of the heights of all Year 11 boys and Year 11 girls from the database of www.censusatschool.co.nz.

While the first comparison is trivial and easily comprehendible for students, the last two need precise mathematical / statistical definition.

**Definition:** If two groups A and B have finite number of elements, the group B tends to be bigger than the group A if the probability of the difference between their elements being positive is more than a half:

\[ P(B_m - A_n > 0) > 0.5 \]  \hspace{1cm} (Inequality 4)

where, \(B_m\) and \(A_n\) represent the values of \(m^{th}\) and \(n^{th}\) elements of the corresponding groups. While this definition is obviously inconvenient for practical use, it gives us a probabilistic basis to derive all the commonly used informal rules (milestone tests) for making a call about two populations. Below we use this definition for comparison of two populations.

**4. Comparison of two populations**

When comparing two populations, we assume that the data in both populations are normally distributed. The fictional examples of the distributions of two populations and their differences are presented on the Figure 2 and 3 below. We use a letter B to represent group of boys and a letter G – group of girls.
As $\mu_D = 20 > 0$, $P(D > 0) > 0.5$, meaning that boys population tends to have longer arm span than the girls population.
In general, if two normally distributed populations A and B have the mean values $\mu_A$ and $\mu_B$ and the mean value of the new population D (where $D = B - A$) is positive ($\mu_D = \mu_B - \mu_A > 0$), then the population B tends to have greater values than the population A as:

$$P(B_m - A_n > 0) = P(0 < B_m - A_n < \mu_D) + 0.5 > 0.5 \quad \text{(see Fig. 3)}$$

**Application 1:** If $\mu_A$ and $\mu_B$ are the mean values of the normally distributed populations A and B, the population B tends to have greater values than the population A if the mean value of the population B is greater than the mean value of the population A, regardless of the spread of the populations

$$\mu_B > \mu_A \quad \text{(Inequality 5)}$$

**Application 2:** To make a call about two normally distributed independent populations all that is needed is to estimate the mean values of the populations using their samples. Note that this is valid for the normally distributed data. It also may be true for some other types of distributions. However, in general, the difference between the means/medians of two populations alone is not enough to make a call.

### 5. Comparison of two populations using single samples

Application of the interval estimate for population means (inequality 3) for the population A and B gives us the 95% confidence interval for each population mean

$$\mu_{\text{Sample B}} - 1.5 \frac{IQR_B}{\sqrt{n}} < \mu_{\text{Population B}} < \mu_{\text{Sample B}} + 1.5 \frac{IQR_B}{\sqrt{n}} \quad \text{(Inequality 6)}$$

$$\mu_{\text{Sample A}} - 1.5 \frac{IQR_A}{\sqrt{n}} < \mu_{\text{Population A}} < \mu_{\text{Sample A}} + 1.5 \frac{IQR_A}{\sqrt{n}} \quad \text{(Inequality 7)}$$

To make a call that “the population B tends to be bigger than the population A” the lower margin of the 95% confidence interval of the population B must be greater than the upper margin of the population A:

$$\mu_{\text{Sample B}} - 1.5 \frac{IQR_B}{\sqrt{n}} > \mu_{\text{Sample A}} + 1.5 \frac{IQR_A}{\sqrt{n}} \quad \text{(Inequality 8)}$$

The inequality (8) leads us to the rule for making a call – if the sample sizes for two populations are the same (n), the population B tends to be bigger than the population A, if

$$\mu_{\text{Sample B}} - \mu_{\text{Sample A}} > 1.5 \frac{IQR_B + IQR_A}{\sqrt{n}} \quad \text{(Inequality 9)}$$
In general, if the sample size for the population A is n and for the population B is m, the population B tends to be bigger than the population A, if

\[ \mu_{\text{sample B}} - \mu_{\text{sample A}} > 1.5 \frac{IQRB}{\sqrt{m}} + 1.5 \frac{IQRA}{\sqrt{n}} \]  

(Inequality 10)

here \( \mu_{\text{sample A}} \) and \( \mu_{\text{sample B}} \) are the mean values of each sample, n and m are sample sizes, IQRA and IQRB are the interquartile ranges of the corresponding samples.

In the next section, based on the inequality 9 we derive and critically evaluate the rules for the informal statistical inference currently adopted in New Zealand high schools.

6. Explaining the rules of the informal inference - “How to make the call”

To derive the rules for the informal statistical inference we consider symmetrical box plots where the median is in the middle between the values of the lower quartile (LQ) and upper quartile (UQ). We also look at the simple relationships between LQ, UQ, IQR (inter quartile range) and median (\( \mu \)):

\[ IQR = UQ - LQ \]  

(Equation 11)

\[ \mu = LQ + \frac{IQR}{2} \]  

(Equation 12)

\[ \mu = UQ - \frac{IQR}{2} \]  

(Equation 13)

6.1. Rule 1 – boxes do not overlap (at any level of NZ curriculum)

If there is no overlap between the boxes of the box and whisker plots of two samples (Figure 4), we can make a call that there is a difference in the population values regardless of the sample sizes (Wild et al., 2011, p.260). Below we show that this rule is true for the sample sizes \( \geq 9 \) only.
Figure 4. If there is no overlap between the boxes of the box and whisker plots of two samples, we can make a call that there is a difference in the population values if the sample sizes are at least 9.

In this case we can see that

\[ LQ_B > UQ_A \quad \text{(Inequality 14)} \]

which leads us to

\[ \mu_B - \mu_A > \frac{IQR_B + IQR_A}{2} \quad \text{(Inequality 15)} \]

The inequality (9) and (15) result in the sample sizes of at least 9:

\[ \frac{IQR_B + IQR_A}{2} \geq 1.5 \frac{IQR_B + IQR_A}{\sqrt{n}} \quad \rightarrow \quad n \geq 9. \quad \text{(Inequality 16)} \]

6.2. Rule 2 – Boxes overlap, but neither median lies inside the other box (age 14, level 5 of NZ curriculum)

If the boxes of samples overlap, but neither median lies inside the other box (Figure 5), we can make a call that the population B tends to be bigger than the population A (Wild et al., 2011, p.260).

Figure 5. If the boxes of two samples overlap, but neither median lies inside the other box, the sample sizes must be at least 36 to make a call that there is a difference in the population values.
We examine this rule below and find out that this is true for the reasonable sample sizes only (n≥ 36). As we can see from the diagram above:

\[ \mu_B - \mu_A > \frac{IQR_B + IQR_A}{4} \]  

(inequality 17)

To make a call about two populations, the condition (9) should be met, which gives us:

\[ \frac{IQR_B + IQR_A}{4} \geq 1.5 \frac{IQR_B + IQR_A}{\sqrt{n}} \implies n \geq 36 \]  

(inequality 18)

Conclusion: if the boxes of two samples overlap, but neither median lies inside the other box, the sample sizes must be at least 36 to make a call that there is a difference in the population values.

In this case Wild et al. (2011, p.260) recommend to restrict sample sizes between 20 and 40. However, we cannot see any reason why the sample size cannot be more than 40.

6.3. Rule 3 - The boxes overlap and each median lies inside the other box (level 5 and 6 of NZ curriculum)

No claim of any kind can be made at the level 5 (age 14) of the curriculum. At the level 6 (age 15) of the curriculum we look at the difference between medians (DBM) and the overall visible spread (OVS).

\[ \text{DBM} = \mu_B - \mu_A \]  

(Equation 19)

\[ \text{OVS} = UQ_B - LQ_A \]  

(Equation 20)

Figure 6. Handy diagram for making a call, commonly used in New Zealand high schools at level 6 (age 15), Wild et al. (2011, p.260).

To derive the rule 3 above empirically developed by Wild et al. (2011, p.260), we used equations / inequalities (9), (11 – 13), (19), and (20). We can make the claim “population B tends to be bigger than the population A” (or vice-versa), if:
\[ DBM > \frac{30\text{OVS}}{\sqrt{n}+3} \]  

(Inequality 21)

where \( n \) is the sample size in each group.

In the table below we provide the comparison of results obtained empirically by Wild et al. (2011, p.260) and derived theoretically in the present paper for the rule 3.

**Table 1.** Comparison of results obtained empirically by Wild et al. (2011, p.260) and derived theoretically in the present paper (Inequality 21)

<table>
<thead>
<tr>
<th>Sample size</th>
<th>Make a call that B tends to be greater than A (or vice-versa) back in population if:</th>
<th>Present article – inequality (21)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wild et al. (2011, p. 260)</td>
<td>Present article – inequality (21)</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>DBM &gt; 33% of OVS</td>
<td>DBM &gt; 35% of OVS</td>
</tr>
<tr>
<td>50</td>
<td>Not available</td>
<td>DBM &gt; 30% of OVS</td>
</tr>
<tr>
<td>100</td>
<td>DBM &gt; 20% of OVS</td>
<td>DBM &gt; 23% of OVS</td>
</tr>
<tr>
<td>1000</td>
<td>DBM &gt; 10% of OVS</td>
<td>DBM &gt; 9% of OVS</td>
</tr>
</tbody>
</table>

6.4. Rule 4 - The boxes overlap and each median lies inside the other box (level 7 of NZ curriculum, age 16)

At the level 7 of the curriculum we make a call that B tends to be greater than A (or vice-versa) back in population if the margins of interval estimates of the population means do not overlap (Wild et al., 2011, p.260). In this article this rule is given by the inequalities (9) or (10).

In this section we derived and explained informal rules of the statistical inference implemented in the NZ high school curriculum. However, we should point out the limitations of these informal rules.

7. Limitations of the Informal Rules of the Statistical Inference
**Limitation 1:** The informal inference rules developed by Wild et al. (2010, 2011) are empirically derived by simulating normal data. Our calculations also are based on the assumption that the data in both populations can be modeled by the normal distribution. Therefore the informal statistical rules currently adopted in New Zealand high schools are limited to normally distributed data.

**Limitation 2:** Our calculations are based on the assumption that the standard deviations of the sample and the population are equal. As all key statistical parameters of the different samples and their populations are different, we do not have any solid reason to suggest, that their standard deviations are the same. On one hand, the standard deviation of a sample can be less than the standard deviation of a population, as the range of the sample is highly likely to be smaller than the range of the population. On the other hand, the standard deviation of a sample increases if the shape of the distribution of the sample is not symmetrical. The range of the sample and the shape of the distribution concurrently affect the value of the standard deviation.

A generated data set of normally distributed Paua shell weights was trialed at Woodford House for the internally assessed achievement standard AS 2.9 (Use statistical methods to make an inference, level 6 of New Zealand curriculum, age 16). Simple random sampling method was used for selecting samples of sizes from 30 to 50. The samples selected by students indicate that about 50% of the samples have the standard deviation less than the population standard deviation. This means, that making a call about two populations has not 5% but more than 5% chance to be incorrect.

**Limitation 3:** For symmetrically distributed data the values of the mean and median are equal. However, even for a sample selected from a symmetrical population the shape of the distribution usually is non-symmetrical and the values of the mean and the median are different. As the median value is not affected by extreme values of the data, preference should be given to the median values. Therefore, in the expressions (9) and (10) $\mu_{\text{sample A}}$ and $\mu_{\text{sample B}}$ should be read as the medians of the appropriate samples.

**Limitation 4:** If all the reasons for limitations 1 to 3 do not exist, i.e. the data in the population and its sample both are normally distributed and the standard deviation of the population and sample are the same, our conclusion about the two populations still has 5% chance to be incorrect as we use 95% confidence interval for estimation of the population means.

8. Summary
8.1. When a comparison of two populations is based on the comparison of their samples, we can make a call that the population B tends to have bigger values than the population A if:

$$\mu_{\text{sample B}} - \mu_{\text{sample A}} > 1.5 \frac{\text{IQR}_B + \text{IQR}_A}{\sqrt{n}}$$

where $\mu_{\text{sample A}}$ and $\mu_{\text{sample B}}$ are the median values and $n$ is the size of each sample, IQRA and IQRB are the interquartile ranges of the corresponding samples.

8.2. If the sample sizes are different, population B tends to have bigger values than the population A if:

$$\mu_{\text{sample B}} - \mu_{\text{sample A}} > 1.5 \frac{\text{IQR}_B}{\sqrt{n}} + 1.5 \frac{\text{IQR}_A}{\sqrt{m}}$$

where $n$ and $m$ are the sample sizes of the population A and B respectively.

8.3. In terms of DBM (difference between medians) and OVS (overall visible spread) population B tends to have bigger values than the population A (or vice versa) if:

$$\text{DBM} > \frac{3\text{OVS}}{\sqrt{n+3}}$$

where $n$ is the size of each sample. This formula can be used for the level 6 and 7 of the NZ curriculum (ages 15 and 16).

8.4. Any call about two populations using single samples has at least 5% chance to be incorrect as we use 95% confidence interval for the population mean estimate.

8.5 There is no significant difference between informal inference rules empirically developed by Wild et al. (2010, 2011) and the informal rules derived in this paper using the formal classical inference techniques. As the informal inference rules developed by Wild et al. (2011, p.260) are empirical rules, they are limited to the particular sample sizes, while the rules provided in this paper can be applied for any sample sizes.

We acknowledge the landmark significance of Wild’s and his colleagues work to deliver the ideas of informal statistical inference to New Zealand high schools. We consider the present paper as a supplement to the work done by Wild et al. (2010, 2011) and Pfannkuch et al. (2010).

8.6. We did not use real data to check the informal rules currently adopted at NZ high schools. We do not argue against the importance of using real data in teaching statistics.
However, we believe that without knowledge of basis of informal inference rules and how to derive these rules, we cannot estimate the limits of their applicability. Otherwise we may develop the wrong impression that these rules can be applied to every real life populations. In fact, these informal inference rules are not valid for all types of distributions. As both the informal rules developed by Wild et al. (2010, 2011) and the rules presented in this paper are based on normal distribution, they may be applicable for some populations only. These rules should be considered as guidelines only for real data and introductory part to distribution free statistical inference.

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Problem Posing, Problem Solving Dynamics
In the Context of
Teaching- Research and Discovery method.

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ABSTRACT

Problem posing is practiced in the context the TR/NYCity methodology of Teaching-Research (Czarnocha, Prabhu 2006), which has been utilized in the mathematics classrooms for a decade. Problem solving turned out to be an essential teaching strategy for developmental mathematics classrooms of Arithmetic and Algebra, where motivation in learning, interest in mathematics, and the relevance of the subject is unclear to adult learners. Problem posing and problem solving are brought into play together so that moments of understanding occur in the context classroom inquiries and discoveries, and a pattern of these moments of understanding can lead to self-directed discovery, becoming the natural mode of learning. Facilitation of student moments of understanding as manifestations of their creative capacity emerges from classroom teaching-research practice and from its relationship with the theory of the Act of Creation (Koestler, 1964). Discovery returns to the remedial mathematics classroom, jumpstarting reform. This Teaching-Research report is based on the collaborative teaching-experiment (Czarnocha et al, 2010) supported by C'IRG grant of CUNY.

Keywords: Discovery, enquiry, problem posing, problem solving, creativity, learning environment, teaching-research.
Introduction: posing the general problem. Enquiry is the path to discovery along which the central problem decomposes into a series of posed questions.

![Diagram](image)

**Fig. 1 Inquiry method of teaching and the decomposition into posed questions/problems.**

That problem posing decomposition is the essential route for reaching discovery; its absence derails success by denying access to that discovery. Transformation of the process of inquiry into a series of smaller posed problems generated by the participants allows every student reach, to discover sought after solution. According to Silver et al (1996), Dunker, (1945) asserted that “problem solving consists of successive reformulations of an initial problem” (p.294) to solve, and that view became increasingly common among researchers studying problem solving. Moreover, (Brown and Walter, 1983) in The Art of Problem Posing, pose and answer the question: “Why, however, would anyone be interested in problem posing in the first place? A partial answer is that problem posing can help students to see a standard topic in a sharper light and enable them to acquire a deeper understanding of it as well. It can also encourage the creation of new ideas derived from any given topic—whether a part of the standard curriculum or otherwise”.(p.169)

The central problem posed in front of the mathematics teacher teaching within an urban community has dimensions that are of a global and a local scale. Both ends of the scale can generate the solution of the problem if appropriate questions are posed to reformulate it to the needed precision for the scale at hand. Such a problem is The Achievement Gap. Thus the central problem addressed in this chapter is how to bridge the Achievement Gap and the role of problem posing/problem solving dynamics in this process. Its two scales are, on the one hand, that which drives political machinery: funding initiatives at the National Science Foundation, ED and other funding agencies, and on the other hand, the situation in a community college mathematics classroom – talent, capacity for deep thinking, yet its clarity disturbed, so grades awarded are not high. The gap at both scales, is just a gap; so that the solution to the common posed problem at one end of the scale, of how to fill/bridge/eliminate the gap, can lead to a flow between the local and the national problem, in that the solution at the local scale informs the problem posed at the global national scale.

The posed problem has multiple dimensions including:
(a) student voices with the actual classroom difficulties, such as: “what is -3 + 5, why
is it not -2”, or “why must I take a long answer test, when the final exam is
multiple choice”, or “why don’t you teach, you just make us solve problems”.

(b) Teacher’s voices with the curricular fixes that they think will/has definitely
eliminated the gap in their own classroom, of say fractions; and who through that
discovery/solved problem, wish to let the secret be available to all students to fix
the fraction gap on a broader scale.

(c) Administration obsessed with standardized exams measuring student skills
development but not their understanding.

The problems posed by the different constituents are sub-probes to the posed problem
of the achievement gap and each of these sub-problems fall into mutually affecting
strands. In the classroom, these fall under the categories of (Barbatis, Prabhu,
Watson, 2012):
(i) Cognition

(ii) Affect

(iii) Self-Regulated Learning practices.

In this article we will illustrate our classrooms’ problem posing possibilities.
Mathematics is thinking technology in which posing problems, attempting to solve them,
and solving them to the extent possible with the thinking technology available, is the
foundation and basis of the discipline. By repeatedly posing questions to solve the
problem in its broad scope, we have discovered that creativity, i.p., mathematical
creativity, can jumpstart remedial reform, confirming this way assertions of (Silver et al,
minimal building blocks on which its edifice is constructed. Thus at any level of the
study of mathematics, problem posing and problem solving are inextricable pieces of the
endeavor.

TR/NYCity Model is the classroom investigation of students learning conducted
simultaneously with teaching by the classroom teacher, whose aim is the improvement
of learning in the very same classroom, and beyond (Czarnocha, Prabhu, 2006).
Teaching-Research, NYCity (TR/NYCity) model has been in effective use in
mathematics classrooms of Bronx CC and Hostos CC, the Bronx community colleges
of the City University of New York, for more than a decade. The investigation of
student learning and their mathematical thinking necessitates the design of questions
and tasks that reveal its nature to the classroom teacher-researcher. Thus problem
posing became the method for facilitation of student mathematical thinking employed
by TR/NYCity. This method of teaching naturally connects with the discovery method
proposed originally by (Dewey, 1916) and Moore (Mahavier, 1999) . Utilization of
TR/NYC in conjunction with the discovery method let us, teacher-researchers, to discover that repeatedly posing questions to students facilitates student creativity, and as such it can jumpstart remedial reform in our classrooms (Czarnocha et al, 2010). That realization confirmed the work of (Silver, 1997), (Singer et al, 2011) and others in the field who assert that problem posing is directly related to the facilitation of student creativity.

The Act of Creation by (Koestler, 1964) allows to extend our understanding of classroom creativity to the methodology of TR/NYC itself. The Art of Creation asserts that bisociation – that is the moment of insight is facilitated and can take place only when two or more different frames of discourse or action are present in the activity. Since teaching-research is the integration of two significantly different professional activities, teaching and research, TR/NYC with its constant probing questions to reveal student thinking presents itself as the natural facilitator of teacher’s creativity as well. TR Cycle below is the theoretical framework within which problem posing/problem solving dynamics as the terrain of student and teachers classroom creativity is being iterated through consecutive semesters. The process of iteration produces new knowledge about learning and problem posing/problem solving instructional materials.

Fig. 2 Teaching Research Cycle with two iterations.

(← - → first cycle   → second cycle)
The TR cycle, iterated every semester of teaching the particular course, allows to diagnose student difficulties at any moment of learning, to design appropriate instruction and to assess its effectiveness. During each semester, student difficulties are cycled over at least twice so that the diagnosed difficulty can be addressed and its success assessed in agreement with the principles of adaptive instruction (Daro et al, 2011, p28). Over the span of several semesters, the methodology creates an increasing set of materials which are refined over succeeding cycles, and acquire characteristics of use to all students studying the mathematical topics under consideration. The learning environment itself develops into a translatable syllabus for the course from several supports for the learning in the classroom so that the applicability of the techniques at use in a teaching-research (TR-NYC) classroom, is easy and replicable for other instructors facing the same difficulties of the achievement gap in their own classroom, and who have an interest in becoming a teacher-researcher in the process of finding the solution to the larger problem in their own classroom.

In the classes of Remedial Mathematics (i.e., classes of Arithmetic and Elementary Algebra) at the community college, Teaching-Research Experiments have been carried out since 2006. In the period from 2006-2012, success began to be evidenced in 2010 following a broader teaching-research team approach described later in this section. The initiative in Remedial Mathematics, followed the successful use of the methodology in calculus classes under the NSF-ROLE#0126141 award, entitled, Introducing Indivisibles in Calculus Instruction. In Calculus classes (NSF-ROLE#0126141) when the appropriate scaffolding dynamic had been embedded in the Learning Environment, students underprepared in, to name the main difficulties, fractions on the line, logic of if-then, algebra of functions and Limit (essential for definite integral conception as the limit of the sequence of partial Riemann sums) were nonetheless able to perform at an introductory Analysis level (as distinct from the level of standard calculus course). Discovery was the ‘natural’ means of exploration in calculus classes, and enquiry leading to discovery through problem posing/problem solving dynamics was able to take place without student resistance.

In classes of Remedial Mathematics, the situation though, is markedly different. Student resistance to learning is prompted by years of not succeeding in the subject, and the general attitude is of ‘just tell me how to do it’. Discovery and enquiry are not welcome means. In the period 2006-2010, the mathematical materials continued to develop, and the learning trajectory of fraction described later also developed in this period. However, the success was not in student learning. In 2007-2008, as part of CUNY funded teaching experiment, Investigating Effectiveness of Fraction Grid, Fraction Domino in mathematics classrooms of community colleges of the Bronx, it was already discovered that a satisfactory student partnership in learning, a didactic contract (Brousseau, 1997) or in classroom language, a mutual “handshake” confirming the commitment to student learning, was essential in confirming the role of problem posing on the affect and self-regulatory learning (Akay and Boz, 2010). In 2010, following a Bronx Community College consultancy to FET colleges in South Africa, a new direction to address the...
problem was found. The situation in classrooms whether in South Africa or in the Bronx, needed a simultaneous attention to student affect.

**Development of Learning Environment.** The relationship between cognitive and affective components of learning has recently been recognized (Araujo et al, 2003; Gomez-Chacon, 2000). According to (Goldin, 2002,) "When individuals are doing mathematics, the affective system is not merely auxiliary to cognition - it is central“p.60. (Furinghetti and Morselli, 2004) assert, (in the context of the discussion of mathematical proof) "The cognitive pathway towards the final proof presents stops, dead ends, impasses, steps forward. The causes of these diversions reside only partially in the domain of cognition; they are also in the domain of the affect." (p.217). There is a need, in addition to attention being paid to the cognitive pathways, to consider [and impact] the affective pathways, which are described by (DeBellis and Goldin 1997) as „the sequence of (local) states and feelings, possibly quite complex, that interact with cognitive representation (p.211).

A learning environment began to develop under iterative loops of the TR cycle, and the components of this learning environment are captured in the concept map below. The teaching-research team now constituted a counselor (also the Vice President for Student Development), a librarian and the mathematics instructor.
Fig. 3 The components of Learning Environment centered on Creative Problem Solving.

The detailed explanation of how to read the concept map with its emphasis on the improvement of classroom performance as the function of motivation, self-regulated learning and cognitive development, is contained in the Appendix 2.

In the period 2010-2012, during the process of developing the conducive learning environment, three factors emerged as anchoring the learning environment (Barbatis, Prabhu and Watson, 2012), viz., simultaneous attention to:

(i) Cognition (materials and classroom discourse well scaffold, paying attention to the development of the Zone of Proximal Development via meaningful questioning in the classroom and via instructional materials designed in accordance with Bruner’s (Bruner, 1978) theoretical position of development of concepts along the concrete, iconic and symbolic forms)

(ii) Affect (classroom discourse and independent learning guided by development of positive attitude toward mathematics through instances and moments of understanding of enjoyment of problems at hand, extended by self-directed
means of keeping up with the changing attitude toward mathematics and its learning)

(iii) Self-regulated learning practices (SRLp) (learning how to learn, usefulness of careful note-taking, daily attention to homework, to asking questions, paying attention to metacognition and independent work).

Two simultaneous developments took place during the construction of the learning environment anchored in these three aspects. The craft knowledge of the teaching-research team had a common focus of employment – the development and viewing of the mathematical material on several planes of reference (Koestler, 1964), i.e., for a problem, say $\frac{1}{2} + \frac{1}{3}$, the counselor of the mathematician-counselor pair would keep the mathematical focus constant while alternating between concrete examples of cookies, pizzas, etc; the method exposed students to the process of generalization. This was then extended by the mathematics instructor in removing the monotony of ‘not remembering’ the rules for operations on fractions by using the rules for operations on fractions in more involved problems such as those involving rules of exponents. The novelty and intrigue of decoding the problems of exponents, made the rules for fractions ‘easier’ to remember or look up. Creativity had emerged as an organic development from the craft knowledge of the instructor, however, it was the support of Arthur Koestler’s The Act of Creation (Koestler, 1964) that provided a theoretical base in which to anchor thinking and development of creativity.

**Theory of the Act of Creation.** (Koestler, 1964) sketches the theory of the act of creation, or the creative act and coins the term bisociation to indicate it. Bisociation refers to the ‘flash of insight’ resulting from ‘perceiving reality on several planes at once’ and hence, not just associating two familiar frames, but seeing a new out of them, which had not been possible before. This moment of understanding or bisociation is facilitated in the teaching-research classroom through problem posing which lead to a pattern that changes habit to originality and mathematics is no longer the ‘old and boring stuff that needs to be done’, but is a source of enjoyment, so that even when the class period ends, students are still interested in continuing to puzzle over problems and when the enjoyment translates into performance and closing of the achievement gap a student at a time.

Koestler’s theory of creativity is based on making connections of the concept in question across three domains or shades of creativity: humor, discovery and art. Note that our Creative Learning Environment anchored in Cognition, Affect and SRL assumes overlapping and mutually conducive roles. Humor addresses affect, discovery addresses cognition and learning how to learn when refined so that it is natural, the learner can transform his discoveries to deeper levels, or art. A quick glimpse of Koestler’s theory is encapsulated in the following two concept maps. The Habit and originality concept map provides the workings of the transformation involved in the creative process. The Habit+Matrix = Discovery concept map digs deeper into this transformative process,
showing the important role of affect/humor in the creative process. Both become directly usable in the development of the Creative Learning Environment in the classroom.

Fig. 4 The role of the bisociative act in transforming the habit into originality. Mathematics Teaching-Research though the TR cycle clearly lends itself to creating a problem posing problem solving dynamic. How does it do so? In the next section we provide several actual classroom instances where problem posing has brought back discovery and enquiry on course. The following concept map links the creativity with the Problem Posing/Problem Solving Dynamics.
Fig. 5 The Role of mathematical creativity for the improvement of Learning.

Problem Posing/Problem Solving Dynamics. Problem Posing Illustration 1. This particular example is from an Elementary Algebra class. The time is just after the first exam, about a month into the semester. Students have had shorter quizzes before. On the day from which this example is taken, almost the entire class, stages a rebellion. They state the instructor does not teach and they solve problems and the fact that the class is remedial means the instructor has to teach. A couple of the students explain what they mean by ‘teach’. One student states her previous instructor did a problem on the board and then they did several like it. Another student adamantly declares that she needs “rules” for how to do every problem. After the uproar subsides, the instructor guides
them through the test, throughout ensuring that the student in question is doing the problem – thinking aloud and throughout pointing out the rules or the significant places to pay attention.

**Problem 1** Compute:

(a) $36 + (-20) + 50 - (-17) - 10 = $

(b) $2 - (-4 - 10) $

(c) $-18 - (-6 + 2) $

(d) $2 - (-13) + (-7) - 20 $

(e) $8 - 5 \times 2 + 9 = $

(f) $6 \times 7(-1) - 3 \times 8(-2) $

(m) $7(-4)(8) - 9 \times 6(-2) $

(n) $15 - 2(-5) - (20 - 4) \div 8$

Each problem is solved/thought out aloud by the student selected by the instructor, and she/he reads the problem, and when there is a symbol stated, such as parenthesis, the student is asked for the meaning of the symbol (posing a problem). Once the whole problem is read aloud with meaning, the student has to determine which is the order in which to proceed and why (solving a problem), and then the student actually does the computation in question.

At the end of the class, attention is brought back to the work done, how it constituted reading comprehension, paying attention to the structure of the problem and then paying attention to the meaning of individual symbols and thinking of structure and meaning together. There was clarity, satisfaction, and a turnaround in problem solving after this session.

What did this session do in the classroom? First, it debunked the myth, that one has to memorize something in order to solve every problem. Second, it took away the authority of the teacher as knowing which the class was reluctant to give up, and finally when each person carefully read and translated/made sense of the problem in terms of symbols and structure, they saw the process of posing and solving working in unison with one of their own classmates carrying out all the thinking. Hence, for example, when the student who was doing the problem, read “parenthesis”, she was questioned, as to what the parenthesis means, and what role it had to play in the problem (posing problems). The mathematical language with its various hidden symbols, many symbols with one meaning, or one symbol with many meanings are all sources of ‘confusion’ for students and situations such as the one narrated here, provide for self-reflection, and clarification of the language.
and the meaning of the language of mathematics. It requires many posed questions along the way to clarity. Note, how affect, cognition and metacognition, all three enter the dialogic thinking that instructor and students went through together.

**Problem Posing Illustration 2.** In this example, the class is Elementary Algebra. Students have trouble determining which rule of exponents is to be applied for the given problem. There is a tendency to arbitrarily use anything without justification. The class problems are followed by a quiz, in which there is great difficulty among students in determining which rule is applicable for the problem under consideration. Again, it is a question of not being able to slow down the thinking to observe the structure of the problem and the similarity of the structure with one or more rules. Students are asked to work on the following assignment:

Rules of Exponents
1. \( a^n \times a^m = a^{n+m} \)
2. \( \frac{a^n}{a^m} = a^{n-m} \)
3. \( (a^n)^m = a^{nm} \)
4. \( a^0 = 1 \)
5. \( a^{-n} = \frac{1}{a^n} \)

Make up your own problems using combinations below of the rules of exponents:
- Rules 1 and 2
- Rules 1 and 3
- Rules 1, 2 and 3
- Rules 1 and 4
- Rules 2 and 4
- Rules 1, 2 and 5
- Rules 1 and 5
- Rules 1, 2, 3, 4, and 5

Solve each of the problems you created.

It was noted in the work students submitted that they created problems that had one term that required the use of say Rule 1 (e.g. \( X^7 Y^8 \)) and another term that required the use of Rule 2 (e.g. \( \frac{x^5}{x^3} \)), but there were no problems that had one term requiring the use of both rules (e.g. \( \frac{y^5 \cdot y^7}{y^{10}} \)). This gives the instructor in question a point from which to further develop the problem solving through deeper problem posing, i.e., through dialogic think aloud face to face sessions, students are asked to observe the structure of the given problem and state the similarity to all rules where similarity is observed (this led to examples of posed questions which led to making complex exercises by the teacher-
researcher). This increases student repertoire in problem solving as evidenced in the following quiz and test.

**Problem Posing Illustration 3.** In this illustration, we provide the triptych used in Statistics classes (also used in Arithmetic and Algebra, but not included here), developed through Koestler’s work of the development of creativity. A triptych in Koestler’s usage is a collection of rows as shown below where the columns indicate humor discovery and art. In order to get to the discovery of the central concept, the learner can work their way into probing the concept through some word that is known and even funny. Students are provided the triptych below with two rows completed. These completed rows are discussed in class as to whether they make sense. Students clarify their understandings in the discussion. It is then expected that students will complete all rows of the triptych and write a couple of sentences of explanation of the connections between the 3 words. When all students have submitted their triptychs, the class triptychs are placed on the electronic platform, Blackboard, and students view and reflect on each others’ work, and create a new triptych for the end of the semester again with a few sentences explaining the connections of the concept and its illustration across the row of the triptych.

**Statistics Triptych**

Trailblazer ↔ Outlier ↔ Original/ity

↔ Sampling ↔

↔ Probability ↔

↔ Confidence interval ↔

↔ Law of large numbers ↔

Lurking variable ↔ Correlation ↔ Causation

**The Algebra and Arithmetic Triptych**

These classes need greater scaffolding with the triptych and here the elements of the triptych are introduced Just-in-Time as the topic under consideration is being covered in the class. Hence for example:

Powers ↔ decimal representation ↔ polynomial is discussed during the chapter on polynomials. The following is the general strategy of facilitation of discovery and understanding from the teacher-researcher’s perspective:
Problem posing is a constant in the discovery oriented enquiry-based learning environment. Operations on integers and i.p., adding and subtracting with visualizing of the number line forms the basis for ongoing questioning and posing of a problem between students and teacher-researcher.

Algebra as the field of making sense of structure simultaneously with making sense of number provides opportunities for problem posing along the Particularity $\leftrightarrow$ Abstraction $\leftrightarrow$ Generality of the Arithmetic-Algebra spectrum. In Algebra classes, it is harder to ease use of scaffolding, and problem posing occurs solely on the side of the teaching-research team in wondering about the way to include triptychs in the Learning Environment mix. In the process, the triptych rows evolve into ‘simpler’ usable forms.
Discussion and Results. The results discussed below were obtained after 3 TR cycles for this work. They will be incorporated into the next TR cycle based on the described ideas and practice. We have discussed how our cyclical involvement in TR-NYC model of teaching research to solve the problem of our classrooms – students’ understanding and mastery of mathematics led us to pose to ourselves a general question: what are the necessary components of student success in mathematics? Our answer to this problem investigated in the teaching experiment Jumpstart to Reform directed our attention to student creativity as the motivating factor for their advancement in learning. In turn, our facilitation of student creativity is scaffolded by a series of posed problems/questions designed either by the teacher or students of the classroom. The quantitative results (Appendix 1) of the teaching experiment Problem Solving in Remedial Mathematics – Jumpstarting the Reform supported by C³IRG 7 awarded to the team in 2010 confirm the impact of the approach for the improvement of student problem solving capacity. We point out that the art of posing series of problems scaffolding student understanding strongly depends on teachers’ judgment concerning the appropriate amount of cognitive challenge. Solving these problems in practice leads again to posing of a general question, which, in agreement with the principles of TR/NYCity leads beyond the confines of our classroom: What is a learning trajectory (LT) of, for example, fraction in my classes? We illustrate the learning trajectory for fractions that developed over the period 2006-2012, with some movement at times, none at others and a lot more when students are active learners. Problem posing has been an active element in that process within the student - teacher mutual understanding.

(a) Meaning of fraction established and revisited
(b) How to visualize fractions? Fractions Grid developed as a visual tool (Czarnocha, 2008)

Over time the following LT developed:

![Learning Trajectory for fractions](image-url)
(i) Equivalent fractions visualized – operation: scaling – visualize with FG and then scaling
(ii) Increasing, decreasing order arrangement- prime factorization – common denominator – FG and then reasoning; common denominators are meaningful before any other standard operations
(iii) Addition and subtraction
(iv) Multiplication
(v) Division
(vi) Transition to language – what is half of 16?

(c) Proportional reasoning: Picture in various versions – seeing the interconnectedness of fraction in different representations: decimal, percent, pie chart
(d) Meaning of fraction revisited.

This learning trajectory will be refined through subsequent cycles of the course. Developing the LT for fractions is an illustration of how problem posing works in the context of a satisfactory handshake on the part of learners. Problems utilizing exponents is an example of active problem posing leading to its successful integration by learners. The mastery of the language of mathematics through self-directed attention to reading comprehension is an example of how the repertoire needed for problem posing and solving needs to be consistently built up.

The development of several Learning Trajectories one of which is shown here demonstrate the usability of the methodology and developed materials for a much larger audience of students who fall in the category of self-proclaimed "no good at math" "dont like math" etc. The process of development of Learning Trajectories proceeds through the elimination of learning difficulties in the collaborating classrooms.

Repeated problem posing-problem solving dynamic increases learners’ repertoire of recognizing own moments of understanding and the emerging pattern of understanding. Writing as the medium utilized for learning to write and writing to learn, makes the understanding lasting, concrete and reusable by learners.

The overarching result is that a discovery-based approach to the learning of Basic Mathematics coupled with due attention to cultivation of positive affect is found to sustain development of learning “how to learn”. The learning environment so created is thus a creative learning environment in that it is capable of stimulating creative moments of understanding and extending these to pattern of understanding that transforms learners' habits of doing/learning mathematics to an enquiry oriented approach that fosters enjoyment and consequently boosts performance. Students’ didactic contract/handshake toward their own learning markedly improves in having found mathematics to be enjoyable and the success in tests boosts confidence and wanting to achieve. Fear with which the class starts the semester and the accompanying it resistance to learning are non-existent in the majority of the class and the two that continue to hold on to it are a
minority and begin taking greater interest. The emphasis on classroom creativity outlines the pathway across the Achievement Gap.

**Conclusion.** Mathematics as the creative expression of the human mind, is intrinsically questioning/wondering why and how, and through reflection/contemplation, gaining insight through careful justification on the answers to the questions posed. Problem posing and problem solving are thus the core elements of ‘doing mathematics’. In contemporary contexts of teaching and learning of mathematics, this core of mathematics, is hidden from sight, and a syllabus, learning objectives, learning outcomes, etc., are more prominent, making mathematics seem like a set of objectives and sometimes even called skills to be mastered by the student who is then considered proficient or competent in those skills. The high failure rate in mathematics starting as early as third grade (MSP-Promyse, 2007), dislike of mathematics reflected not just among students, but societally, the low number of students seeking advanced degrees in mathematics are reflective of mathematics not being appreciated for what it is – the quest of the human mind toward knowing, and wanting to know why and how.

In the particular context of community colleges of the Bronx of the City University of New York, and analogously the large percentage of high school students who need remedial/developmental mathematics courses in college, problem posing has to be directly connected and on a regular basis with the classroom curriculum. The objective is urgent: closing the achievement gap. The problem as it exists, is that absence of proficiency in mathematics (i.e., scores on placement tests) could well prevent students from college education. The question is how to change this trend? (Knott, 2010) states, “Recent developments in mathematics education research have shown that creating active classrooms, posing and solving cognitively challenging problems, promoting reflection, metacognition and facilitating broad ranging discussions, enhances students’ understanding of mathematics at all levels. The associated discourse is enabled not only by the teacher’s expertise in the content area, but also by what the teacher says, what kind of questions the teacher asks, and what kind of responses and participation the teacher expects and negotiates with the students. Teacher expectations are reflected in the social and socio-mathematical norms established in the classroom.” (Vygotsky, 1978.) describes the Zone of Proximal Development (ZPD) as “the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or collaboration of more capable peers” P.86. In the classroom environments we encounter, where to be effective, the classroom environment requires a careful integration of simultaneous attention to cognition, affect and self-regulatory learning practices (Prabhu, Barbatis, Watson, 2012), the ZPD has to be “characterized from both cognitive and affective perspectives. From the cognitive perspective we say that material should not be too difficult or easy. From the affective perspective we say that the learner
should avoid the extremes of being bored and being confused and frustrated (p.370)” (Murray & Arroyo, 2002).

Teaching and learning in a teaching-research environment is necessarily collaborative, as our work has demonstrated. This collaborative nature creates an open community environment in the classroom, which is beneficial to the problem posing requirement. Mathematics as enquiry, as enjoyment and as development of the thinking technology do not remain terms or unfamiliar notions to learners. In the span of one semester, college readiness has to be achieved so that the regular credit bearing mathematics courses can be completed satisfactorily. Enquiry facilitating discovery, the modus operandi, now possible to take hold because of the creative learning environment, has provided learners with the keys to success in learning and understanding of mathematics.

The problem posing style of education in general whether Freire (Freire, 2000), “reading the world”, or in the style of Montessori (Montessori, 1972), in the design of the learning environment, all find use and applicability in the Remedial Mathematics classroom. Further, the Discovery method, or Moore method as learned from William Mahavier, Emory University, (Mahavier, 1999) successful in calculus classes, now finds a route into generating learners enjoying and performing well in mathematics in Remedial classes, leading the route to the closing of the achievement gap and creating readiness for higher level mathematics classes.

References


Appendix 1
Problem Solving in Remedial Mathematics: A Jumpstart to Reform.
William Baker, Bronislaw Czarnocha, Olen Dias and Vrunda Prabhu
Short Summary and report from the project

Implementation
Four professors were involved in the TR experiment at two different community colleges in an urban environment Professors: Prabhu, Dias, Baker and Czarnocha. The control groups (class sections) used a more traditional curriculum of strictly class lecture organized by topics: whole numbers, fractions, decimals, ratio and proportions and percents. The experimental group used modified discovery learning and modules focused on problem solving. Each instructor taught one experimental and one control section.

Statistical Analysis of the results
A pre-test (5 questions) was administered at the beginning of the semester to students and a post-test (8 questions) at the end to students in all sections of both the control and experimental sections, there were 4 common questions to both exams. These tests focused on students’ abstract structural (Sfard, 1991) understanding of the second strategy phase of Polya’s stages of problem solving more specifically the transition from initial reading to strategy formation. In the control group there were 46 students that completed both the pre- and post-tests while in the experimental group there were 34. The mean score of the control group on the pre test was 43% and on the post-test it was 42% the mean score of the experimental group on the pre-test was 40.1% and on the post-test it was 54.5%.

Pre & Post Results (all instructors)
Control group: (N=46)

<table>
<thead>
<tr>
<th></th>
<th>Pre</th>
<th>Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Score</td>
<td>43%</td>
<td>42%</td>
</tr>
</tbody>
</table>

Experimental group (N=34)

<table>
<thead>
<tr>
<th></th>
<th>Pre</th>
<th>Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Score</td>
<td>40%</td>
<td>54.7%</td>
</tr>
</tbody>
</table>

The four common or repeated exercises are listed individually for the experimental group (N=34)

<table>
<thead>
<tr>
<th>Exercise</th>
<th>Pre</th>
<th>Post</th>
<th>p-value</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exercise #1) Ratio</td>
<td>33.8%</td>
<td>46.3%</td>
<td>0.1</td>
<td>not significant</td>
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<tr>
<td>Exercise #2) Division</td>
<td>63.2%</td>
<td>61.7%</td>
<td>0.3</td>
<td>not significant</td>
</tr>
<tr>
<td>Exercise #3) Operator</td>
<td>38.2%</td>
<td>61.7%</td>
<td>0.04</td>
<td>significant</td>
</tr>
<tr>
<td>Exercise #4) Quotient</td>
<td>58.9%</td>
<td>79.4%</td>
<td>0.05</td>
<td>borderline significant</td>
</tr>
</tbody>
</table>

Significance is taken to be p<0.05.
The values of the pretest for the control (43%) and experimental groups (40%) were not significantly different (p=0.78) and thus, we fail to reject the null hypothesis that the pre and post test means are. In contrast, for the experimental group (p=0.0001) we reject the null hypothesis. Thus, we conclude that, the mean scores for the pre and post tests are significantly different which demonstrates a substantial improvement in the experimental group students’ ability to think about problem solving in an algebraic or structural manner.

The difference between the control (42%) and experimental (54.7%) mean scores on the post test was significant at the 0.01 level (p< 0.001). Thus, we reject the null hypothesis that the means for these two groups are the same and instead conclude that the mean score for the experimental group was significantly higher. The difference between the control group pre-test (43%) and the experimental group (40%) was not significantly different (p=0.48). However, the mean score for the control group post-test (42%) was significantly different than the experimental group post- test (54.7%, p=0.02). That the difference between the pre test scores for the control and experimental groups were not significantly different but the post test scores were indicates that while both groups started at the same level of ability with structural problem solving the experimental group outperformed the control group by the end of the semester.

This indicates that the problem solving module approach implemented through guided discovery learning had a real impact on students’ ability to think about problem solving in a structural manner that would allow them to understand solution strategies for typical word problems involving concepts of: ratios, quotients, operator-fraction as well as percent.

Appendix 2. How to read a concept map?
First we look at the general structure of the concept map, where we find four components of learning environment: Creative Problem Solving set, student Success Manual, Making Sense of Numbers – a component of pedagogy, and the collection of instructional materials Story of Number. Each of those components have branches going up explaining the content or the method utilized by each, and they have branches going down, which shows the relation the particular component makes with Cognition, Affect and Self-regulatory learning which found themselves to be necessary student success in mathematics.
Second, we can investigate branches of each component separately paying attention to particular concepts (in boxes) and to connecting phrases (no boxes). We can proceed down or up along the branches, depending on preference. For example we see that the Creative Problem Set connects cognition with affect by facilitating enjoyment in problem solving and thus in learning mathematics, which leads to the improved performance.